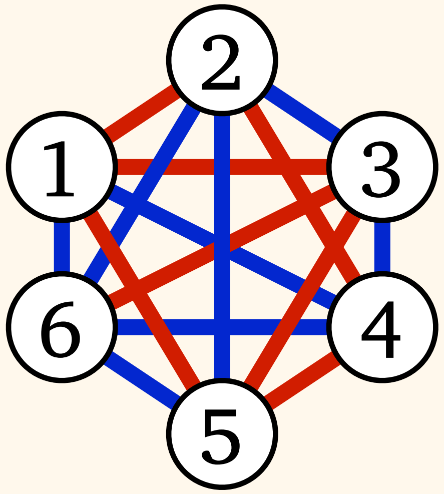
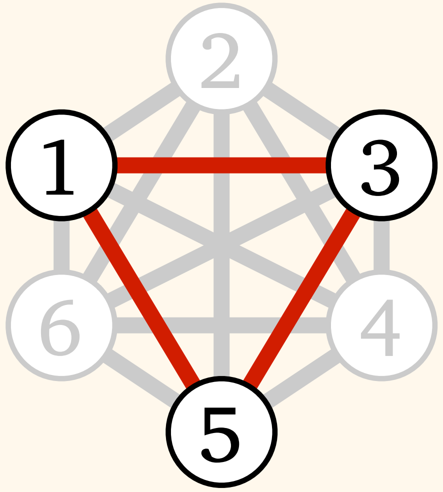
Ramsey Theory

DDA Course

week 6





ON A PROBLEM OF FORMAL LOGIC

By F. P. RAMSEY.

[Received 28 November, 1928.—Read 13 December, 1928.]

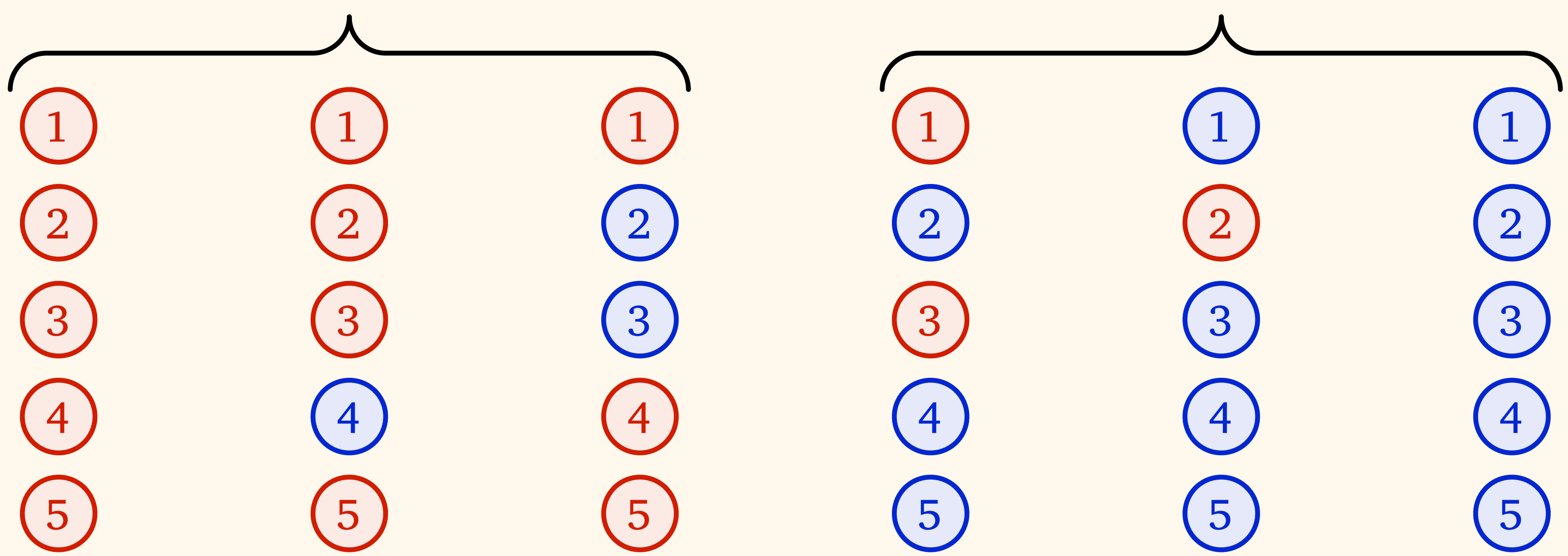
This paper is primarily concerned with a special case of one of the leading problems of mathematical logic, the problem of finding a regular procedure to determine the truth or falsity of any given logical formula*. But in the course of this investigation it is necessary to use certain theorems on combinations which have an independent interest and are most conveniently set out by themselves beforehand.

"… certain theorems on combinations which have an independent interest…"

Pigeonhole Principle

$N = 5$ items, colour each of them red or blue

Always: *at least 3 red* or *at least 3 blue*



Pigeonhole Principle

- Let $n = 3$
- N items, colour each of them red or blue
- If N is large enough, there are always
  - at least n red items or
  - at least n blue items
- Here $N \geq 5$ is sufficient, $N < 5$ is not

Pigeonhole Principle

- Let n be anything
- N items, colour each of them red or blue
- If N is large enough, there are always
  - at least n red items or
  - at least n blue items
- Here $N \geq 2n - 1$ is sufficient

Ramsey Theory

- Generalisation of pigeonhole principle
- Again, we have N items
- However, we will not colour items,
  we will colour ***sets*** of items
  - example: we colour all 2-subsets of items
  - "k-subset" = subset of size k

Ramsey Theory

- Y: set with N items
  - $N = 4$: $Y = \{1, 2, 3, 4\}$
- f: colouring of k-subsets of Y
  - $k = 2$: $f(\{1, 2\}) = \text{red}$, $f(\{1, 3\}) = \text{blue}$, ...
- $X \subseteq Y$ is ***monochromatic*** if
  all k-subsets of X have the same colour

$N = 4, \; Y = \{1, 2, \ldots, N\}, \; k = 2$

Colour each 2-subset of Y:

1, 2 | 1, 3 | 1, 4 | 2, 3 | 2, 4 | 3, 4

{1, 2, 3} is not monochromatic:

1, 2 | 1, 3 | 2, 3

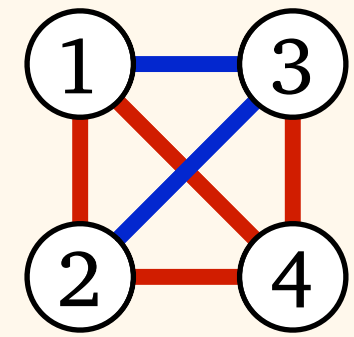
{1, 2, 4} is **monochromatic**:

1, 2 | 1, 4 | 2, 4

$N = 4$, $Y = \{1, 2, \ldots, N\}$, $k = 2$

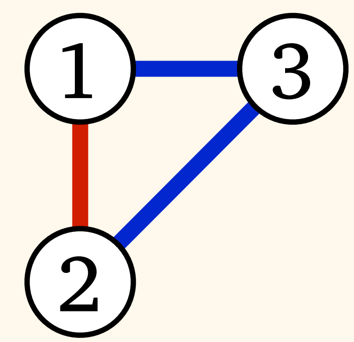
Colour each 2-subset of Y:

1, 2 | 1, 3 | 1, 4 | 2, 3 | 2, 4 | 3, 4



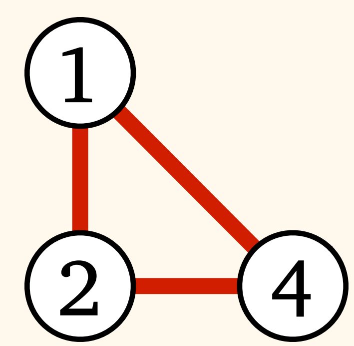
$\{1, 2, 3\}$ is not monochromatic:

1, 2 | 1, 3 | 2, 3



$\{1, 2, 4\}$ is **monochromatic**:

1, 2 | 1, 4 | 2, 4



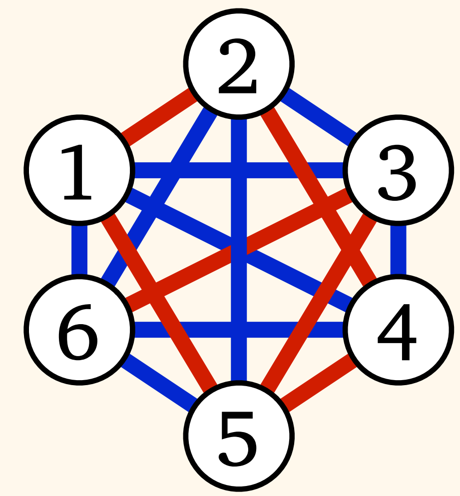
Ramsey Theory

- Let $n = 3, \ k = 2$
- N items, colour each k-subset red or blue
- **Claim**: if N is sufficiently large, there is always a monochromatic subset of size n

$N = 6,\ \ Y = \{1, 2, \ldots, N\},\ \ k = 2$

Colour each 2-subset of Y:

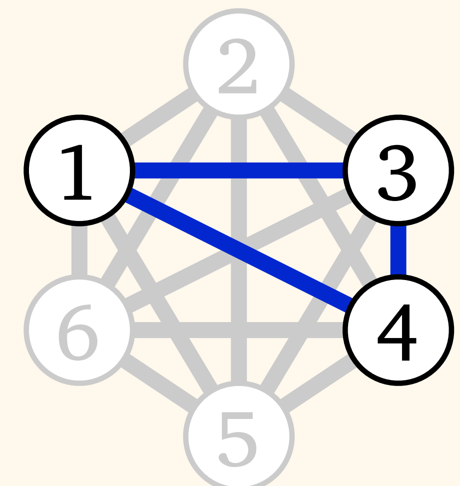
1, 2	1, 3	1, 4	1, 5	1, 6
	2, 3	2, 4	2, 5	2, 6
		3, 4	3, 5	3, 6
			4, 5	4, 6
				5, 6



$N = 6,\ \ Y = \{1, 2, \ldots, N\},\ \ k = 2$

Colour each 2-subset of Y:

1, 2	1, 3	1, 4	1, 5	1, 6
	2, 3	2, 4	2, 5	2, 6
		3, 4	3, 5	3, 6
			4, 5	4, 6
				5, 6

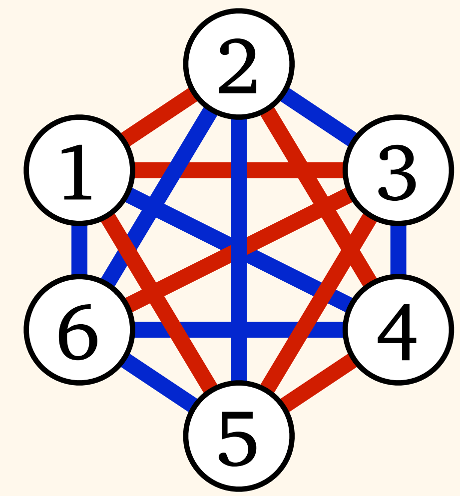


$\{1, 3, 4\}$ is monochromatic

$N = 6$, $Y = \{1, 2, \ldots, N\}$, $k = 2$

Colour each 2-subset of Y:

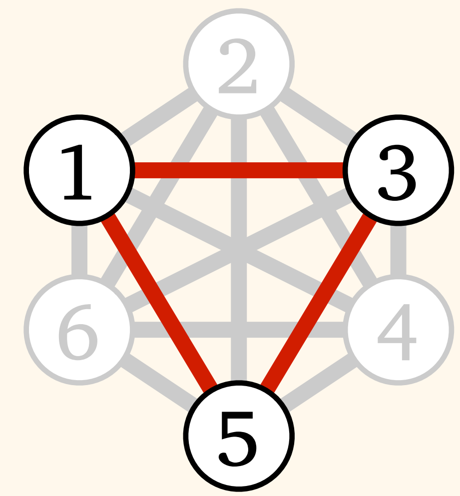
1, 2	1, 3	1, 4	1, 5	1, 6
	2, 3	2, 4	2, 5	2, 6
		3, 4	3, 5	3, 6
			4, 5	4, 6
				5, 6



$N = 6,\ Y = \{1, 2, \ldots, N\},\ k = 2$

Colour each 2-subset of Y:

1, 2	**1, 3**	1, 4	**1, 5**	1, 6
	2, 3	2, 4	2, 5	2, 6
		3, 4	**3, 5**	3, 6
			4, 5	4, 6
				5, 6



$\{1, 3, 5\}$ is monochromatic

Ramsey Theory

- Let $n = 3$, $k = 2$
- N items, colour each k-subset red or blue
- **Claim**: if N is sufficiently large, there is always a monochromatic subset of size n
  - we can show that $N = 6$ is enough
  - we can show that $N = 5$ is not enough

Ramsey Theory

- Let $n = 4, \;\; k = 2$
- N items, colour each k-subset red or blue
- **Claim**: if N is sufficiently large, there is always a monochromatic subset of size n
  - simple upper bound: $N = 20$ is enough
  - a bit more difficult argument: $N = 18$ is enough

Ramsey Theory

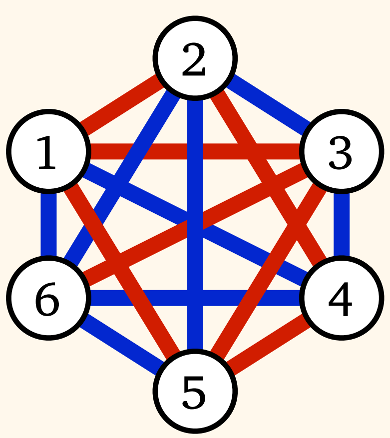
- Let n and k be any positive integers
- N items, colour each k-subset red or blue
- **Claim**: if N is sufficiently large, there is always a monochromatic subset of size n

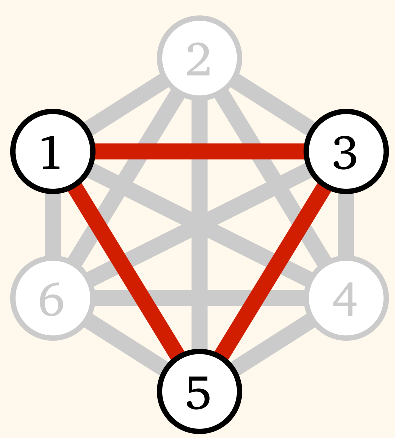
Ramsey Theory

- Let c, n, and k be any positive integers
- N items, colour each k-subset with a colour from $\{1, 2, ..., c\}$
- **Claim**: if N is sufficiently large, there is always a monochromatic subset of size n

Ramsey’s Theorem

- **Theorem**: For all c, n, and k, there is a number $R_c(n; k)$ such that if you take $N \geq R_c(n; k)$ items, and colour each k-subset with one of c colours, there is always a monochromatic n-subset





$R_2(3; 2) = 6$

Ramsey's Theorem

- **Theorem**: For all c, n, and k, there is a number $R_c(n; k)$ such that if you take $N \geq R_c(n; k)$ items, and colour each k-subset with one of c colours, there is always a monochromatic n-subset
    - proof: see the course material
    - numbers $R_c(n; k)$ are called *Ramsey numbers*
    - examples: $R_2(3; 2) = 6$, $R_2(4; 2) = 18$

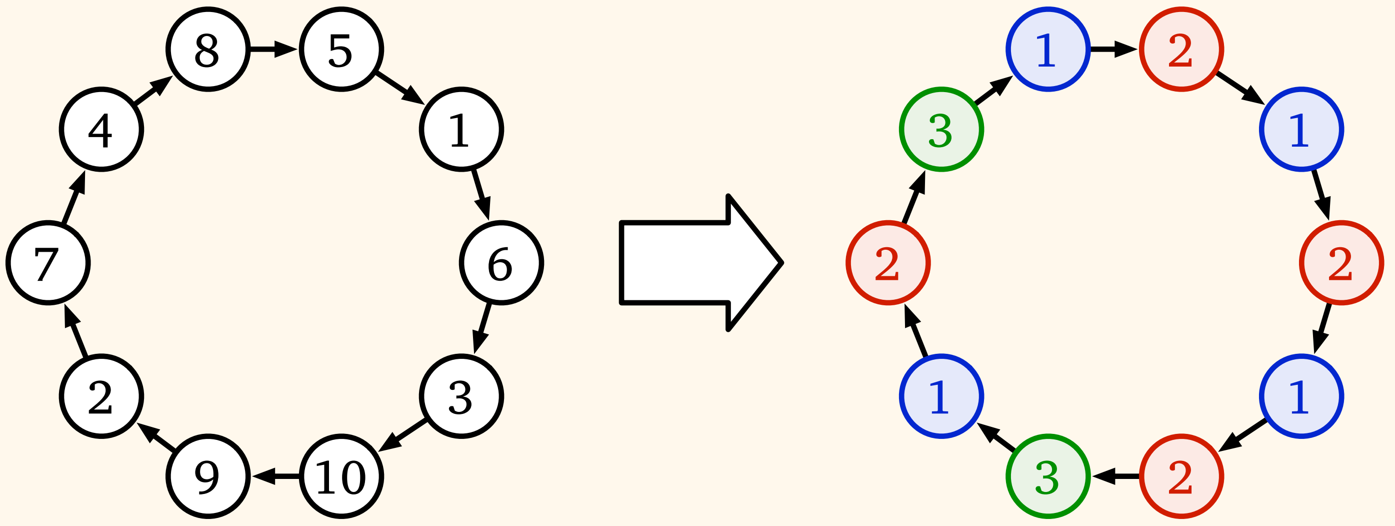
Ramsey’s Theorem

- No matter how you colour subsets, if the base set is large enough, we can always find a monochromatic subset
- Our application: *no constant-time algorithm for 3-colouring directed cycles*
    - no matter how you design your algorithm, if the set of possible identifiers is large enough, we can always find a “bad input”

Colouring in Constant Time?

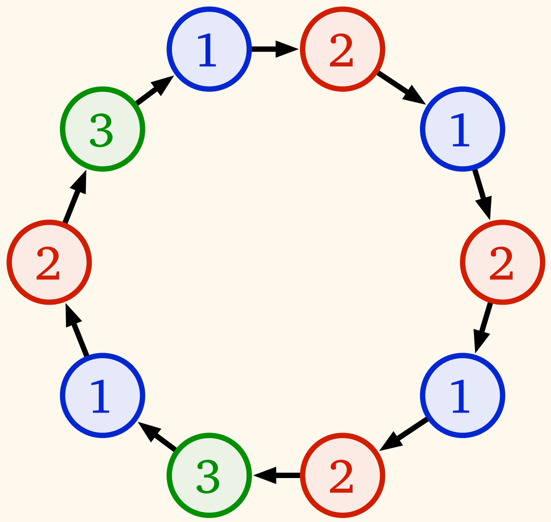
Colouring in Cycles

- Problem: 3-colouring in *directed cycles*
  - unique identifiers from {1, 2, … n}
  - outdegree = indegree = 1



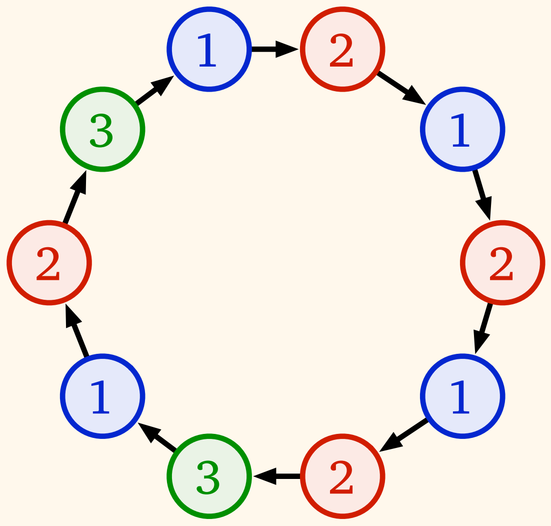
Colouring in Cycles

- Problem: 3-colouring in *directed cycles*
  - unique identifiers from {1, 2, … n}
  - outdegree = indegree = 1
- We know how to solve this problem in time $O(\log^* n)$
  - special case of directed pseudoforests



Colouring in Cycles

- Problem: 3-colouring in *directed cycles*
  - unique identifiers from $\{1, 2, \ldots n\}$
  - outdegree = indegree = 1
- We know how to solve this problem in time $O(\log^* n)$
- Can we do it in time $O(1)$?

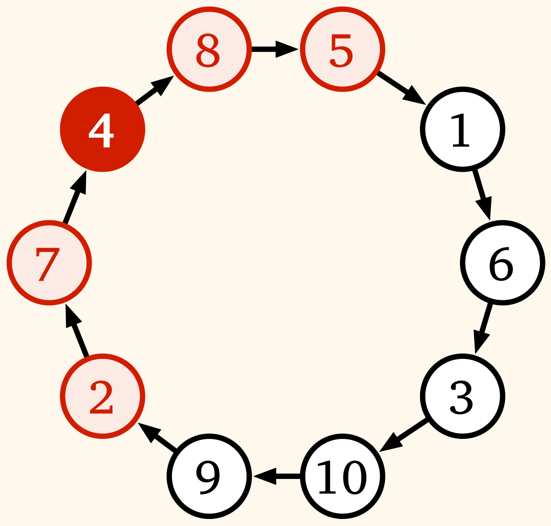


Ramsey Says No

- Assume that algorithm A:
  - in any directed cycle,
    stops in time T for some constant T
  - produces local outputs from {1, 2, 3}
- We will use Ramsey's theorem to show that there is a directed cycle in which A fails to produce a proper vertex colouring

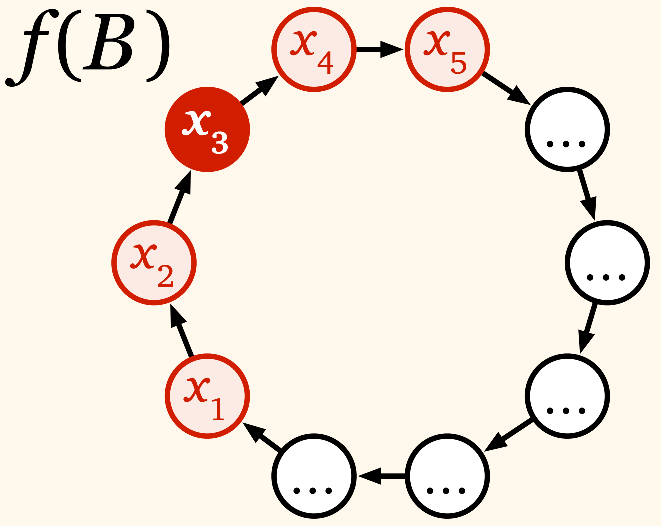
Ramsey Says No

- Example: algorithm runs in time $T = 2$
- Output of a node only depends on $k = 2T + 1 = 5$ nodes around it
  - choose $c = 3$, $n = k + 1 = 6$
  - choose $N \geq R_c(n; k)$
  - c-colour k-subsets of $\{1, 2, \ldots, N\}$: there is a monochromatic n-subset



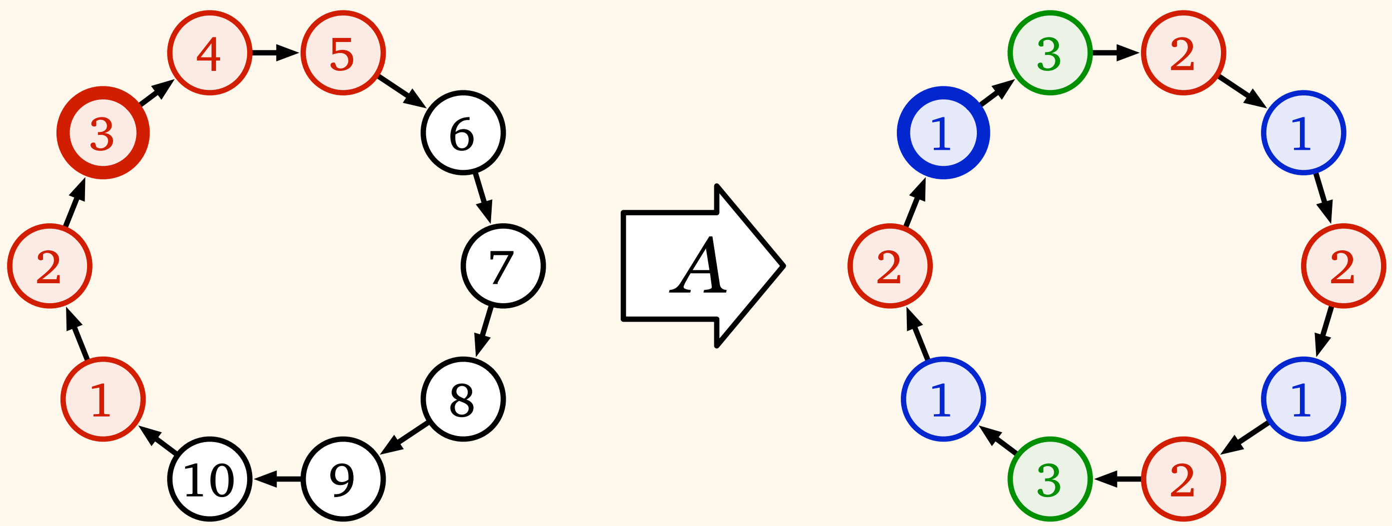
Ramsey Says No

- Set of identifiers: $Y = \{1, 2, \ldots N\}$
- We use algorithm A to colour k-subsets of Y
    - for each set $B = \{x_1, x_2, \ldots, x_k\} \subseteq Y$, $x_1 < x_2 < \ldots < x_k$
    - construct a cycle where nodes $x_1, x_2, \ldots, x_k$ are placed in this order
    - $f(B)$ = output of the middle node



Colour each k-subset of Y:

— what is the colour of {1, 2, 3, 4, 5}?



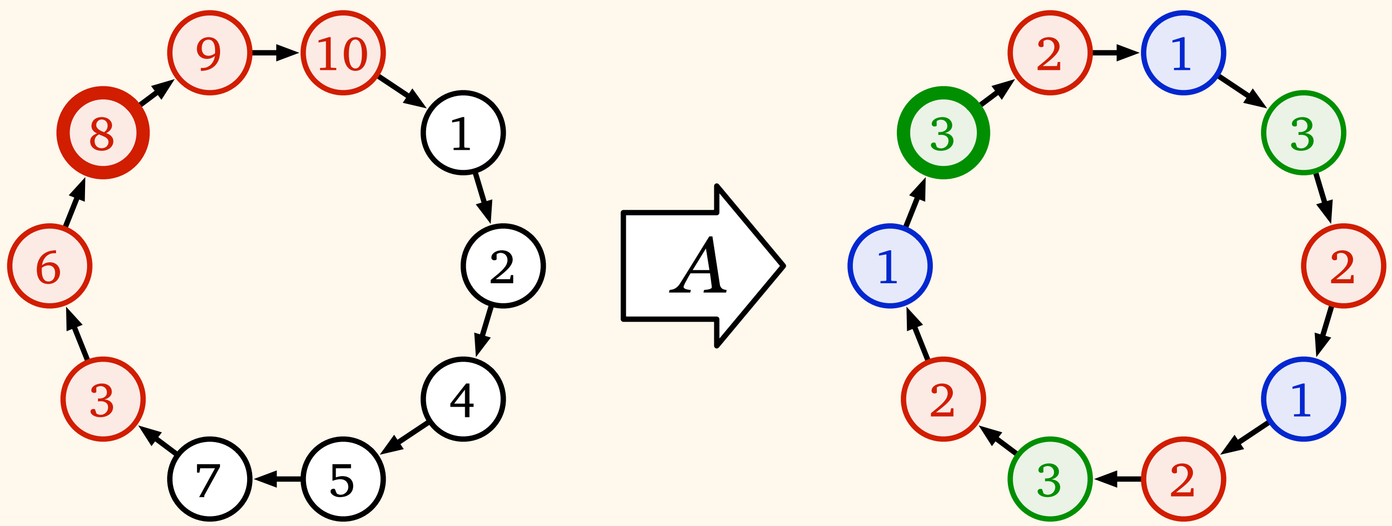
— middle node 3 outputs "blue"

— set $f(\{1, 2, 3, 4, 5\})$ = "blue"

1, 2, 3, 4, 5

Colour each k-subset of Y:

— what is the colour of {3, 6, 8, 9, 10}?



— middle node 8 outputs "green"

— set $f(\{3, 6, 8, 9, 10\})$ = "green"

1, 2, 3, 4, 5 | 3, 6, 8, 9, 10

Ramsey Says No

- We have assigned a colour $f(B) \in \{1, 2, 3\}$ to each k-subset B of Y

1, 2, 3, 4, 5	1, 2, 3, 4, 10	1, 2, 3, 5, 10
1, 2, 3, 4, 6	1, 2, 3, 5, 6	1, 2, 3, 6, 7
1, 2, 3, 4, 7	1, 2, 3, 5, 7	1, 2, 3, 6, 8
1, 2, 3, 4, 8	1, 2, 3, 5, 8	• • •
1, 2, 3, 4, 9	1, 2, 3, 5, 9	6, 7, 8, 9, 10

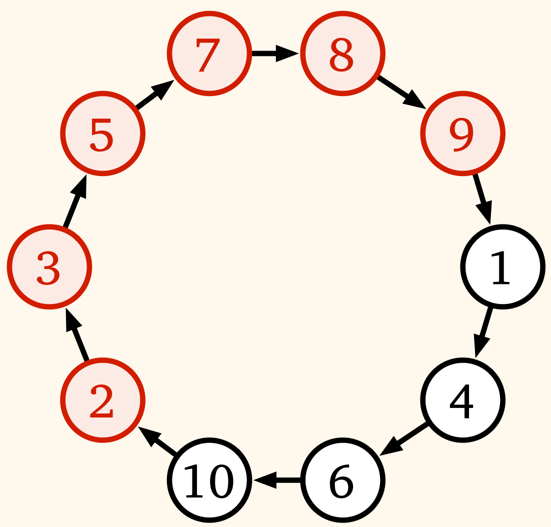
Ramsey Says No

- We have assigned a colour $f(B) \in \{1, 2, 3\}$ to each k-subset B of Y
- Ramsey: set Y was large enough, there is a monochromatic subset of size n
  - example: {2, 3, 5, 7, 8, 9} is monochromatic

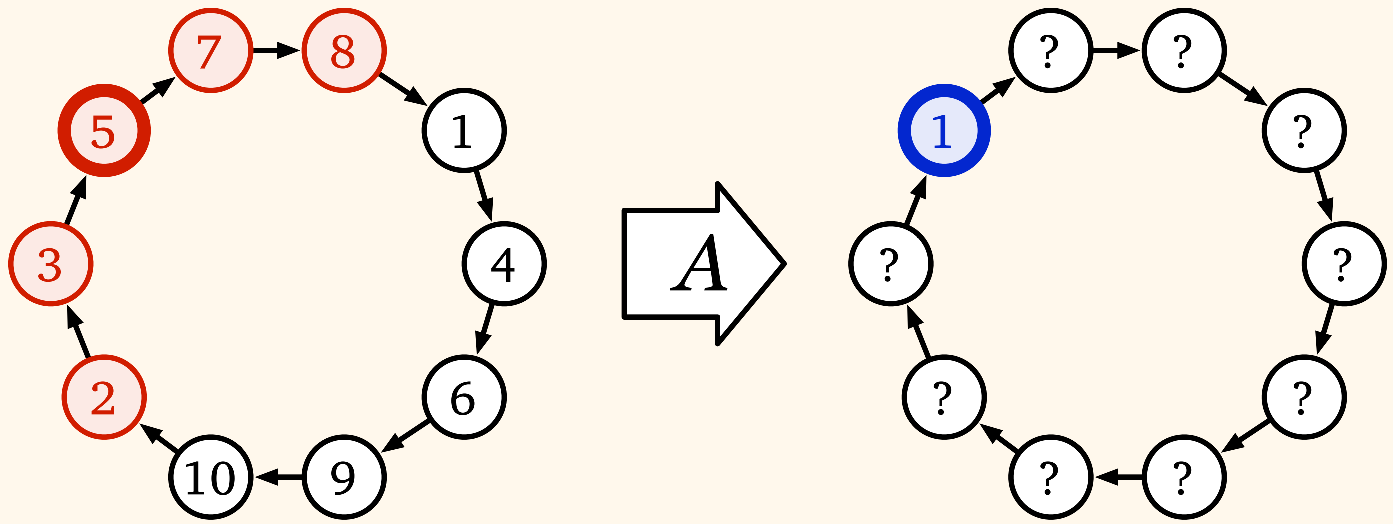
2, 3, 5, 7, 8	2, 3, 5, 8, 9	2, 3, 7, 8, 9
2, 3, 5, 7, 9	2, 5, 7, 8, 9	3, 5, 7, 8, 9

Ramsey Says No

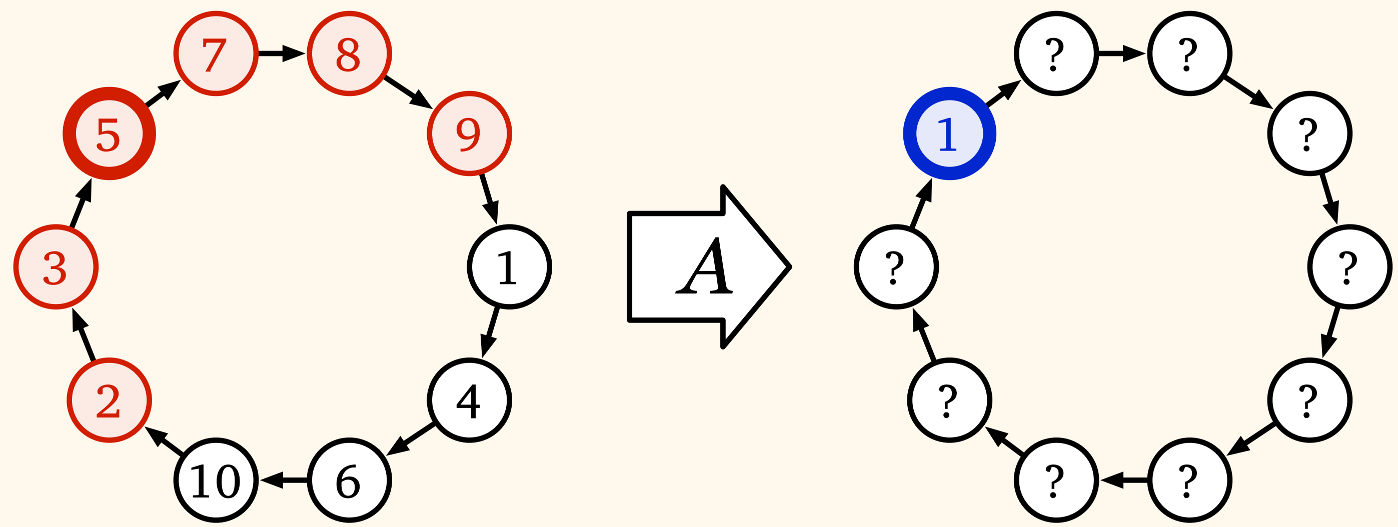
What happens here?



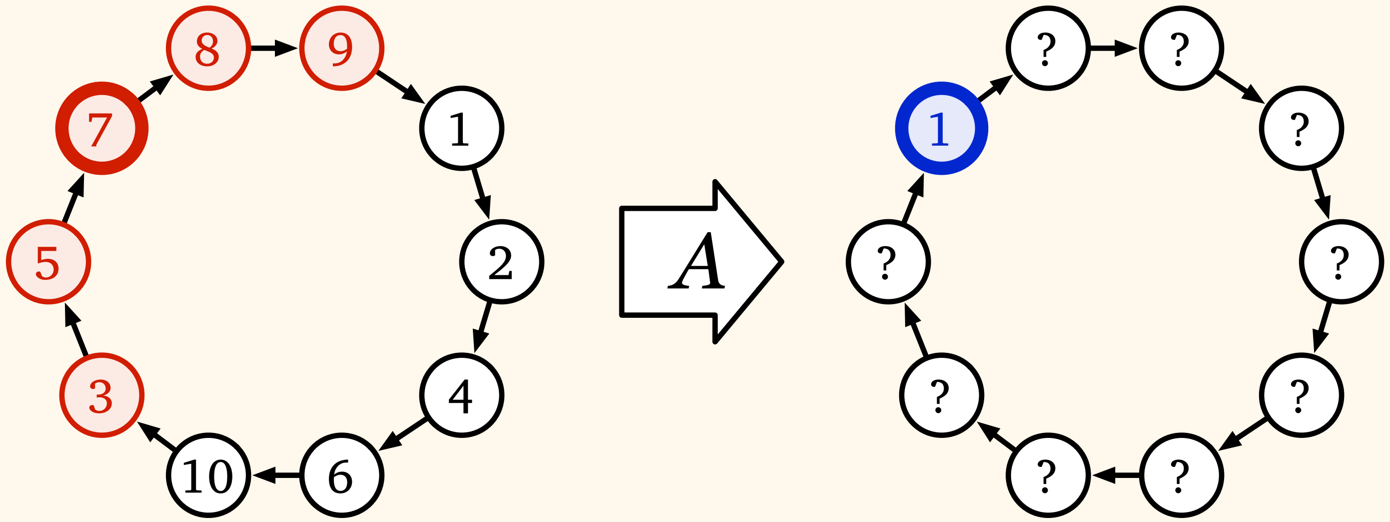
2, 3, 5, 7, 8	2, 3, 5, 8, 9	2, 3, 7, 8, 9
2, 3, 5, 7, 9	2, 5, 7, 8, 9	3, 5, 7, 8, 9



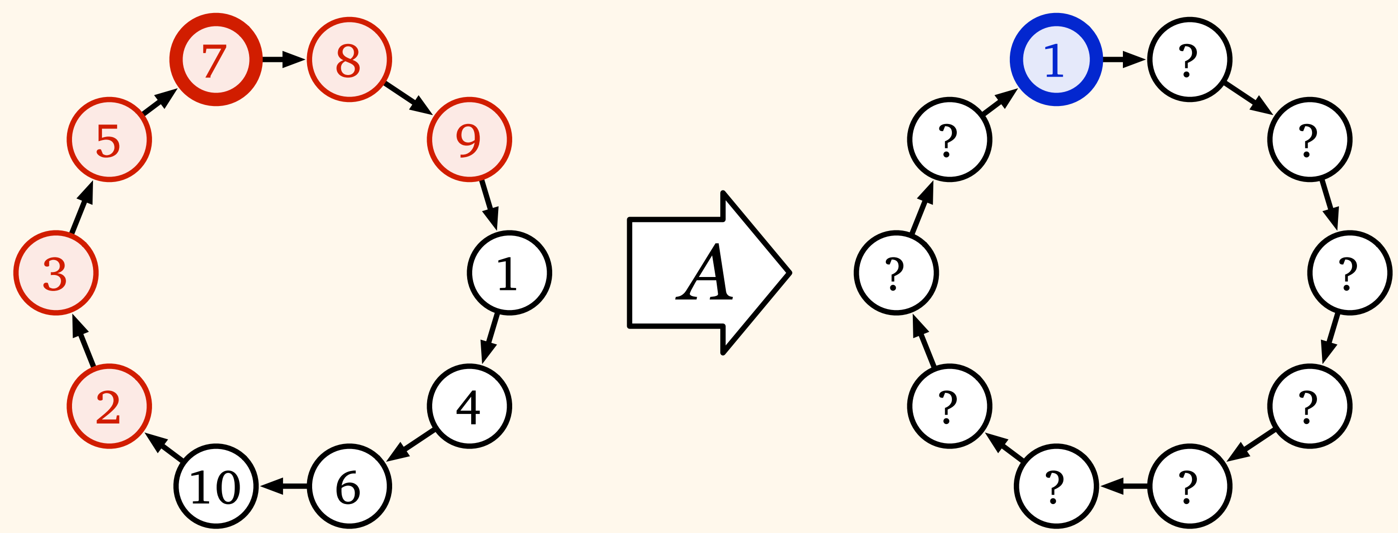
same local neighbourhood, same output



2, 3, 5, 7, 8	2, 3, 5, 8, 9	2, 3, 7, 8, 9
2, 3, 5, 7, 9	2, 5, 7, 8, 9	3, 5, 7, 8, 9



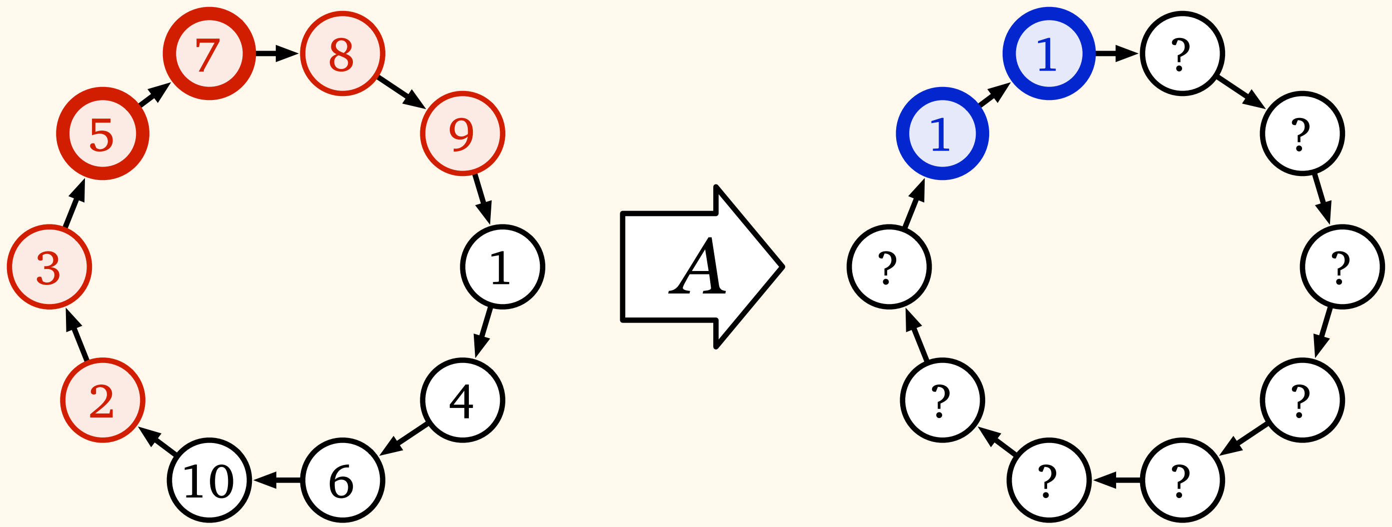
same local neighbourhood, same output



2, 3, 5, 7, 8	2, 3, 5, 8, 9	2, 3, 7, 8, 9
2, 3, 5, 7, 9	2, 5, 7, 8, 9	**3, 5, 7, 8, 9**

Ramsey Says No

Bad output!



2, 3, 5, 7, 8	2, 3, 5, 8, 9	2, 3, 7, 8, 9
2, 3, 5, 7, 9	2, 5, 7, 8, 9	3, 5, 7, 8, 9

Ramsey Says No

- There is no algorithm that finds a 3-colouring in time T
  - the proof holds for any constant T
  - larger $T \rightarrow$ need a (much) larger identifiers space Y

Summary

Distributed Algorithms

- Two models
- Port-numbering model
  - key question: what is computable?
- Unique identifiers
  - key question: what can be computed fast?

Algorithm Design

- *Colouring* is a powerful symmetry-breaking tool
- Port-numbering model
  - bipartite double covers → 2-colouring...
- Unique identifiers
  - identifiers → colouring → colour reduction...

Lower Bounds

- Port-numbering model
  - covering maps
  - local neighbourhoods
- Unique identifiers
  - Ramsey’s theorem
  - local neighbourhoods

That's all.

- Exam: 28 April 2014
    - ***learning objectives!***
- What next?
    - course feedback
    - Master's thesis topics available