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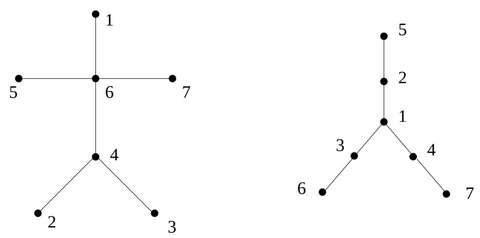
MAT282 Introduction to Discrete Mathematics
Final Exam, Wed 15 Nov. 2000, 8–12 a.m.

1. Let $n \geq r \geq k \geq 1$ be integers. Prove the following binomial coefficient identities:
$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}, \qquad \binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}.$$
Give also a combinatorial interpretation for the second identity.

2. (a) Find the Prüfer codes (as used, e.g., in the proof of Cayley's theorem) of the following trees:



Determine the 8-vertex tree corresponding to Prüfer code (2,7,1,8,2,8).

(b) Prove that every tree T with at least two vertices has chromatic number $\chi(T) = 2$.

(c) Prove that if a tree contains a vertex of degree $d \geq 1$, then it also contains at least d leaves.

3. (a) Consider a graph whose vertices are all the two-element subsets of the four-element basis set $\{1, 2, 3, 4\}$, and there is an edge between vertices X and Y, $X \neq Y$, if and only if $X \cap Y \neq \emptyset$. Draw a picture of this graph, and prove that it is planar.

(b) Prove that the corresponding graph formed out of the two-element subsets of the five-element basis set $\{1, 2, 3, 4, 5\}$ is *not* planar.

4. A tricky situation has developed in the counting of votes in the presidential election of the Republic of A: a total of 370 votes given to Donald Duck (who was not a candidate in the election) should be divided among the presidential candidates B, G, and N, so that candidate B gets at least 200 votes, candidate G gets between 100 and 199 votes, and candidate N gets at most 10 votes. In how many ways can the duck-votes be partitioned among the candidates in the desired manner?

5. Consider the set of strings over the four-letter alphabet $\{A, B, C, D\}$. Denote by a_n the number of n-letter strings that contain an odd number of the letter A. (Thus $a_0 = 0$, $a_1 = 1$, $a_2 = 6$ etc.) Prove that the numbers a_n are related by the recurrence equation $a_{n+1} = 2a_n + 4^n$, and use this recurrence to find a closed-form expression for the numbers a_n. (*Hint:* Partition the $(n+1)$-letter strings according to whether they begin with A or some other letter.)

Grading: Each problem 12 points, total 60 points.