Dense Subgraph Discovery (DSD)

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KDD 2015

Tutorial website

slides and links to relevant papers :

https://densesubgraphdiscovery.wordpress.com/tutorial

can also be found via KDD 2015 website

What this tutorial is about . . .

given a graph (network), static or dynamic

(social network, biological network, information network, . . .)

find a subgraph that . . .

. . . has many edges

. . . is densely connected

why I care?

what does dense mean?

review of main problems, and main algorithms

Outline

- motivating applications
- preliminaries and measures of density
- algorithms for static graphs
- algorithms for dynamic graphs
- problem variants
- conclusions and open problems

Motivating applications

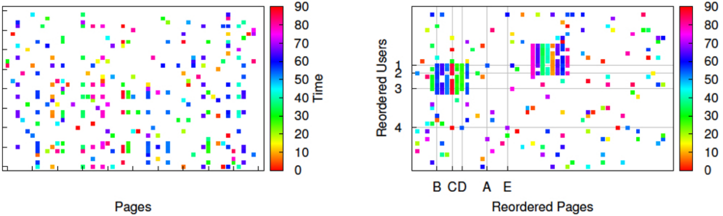
Motivation – correlation mining

correlation mining: a general framework with many applications

- data is converted into a graph
- vertices correspond to entities
- an edge between two entities denotes strong correlation
  1. stock correlation network: data represent stock timeseries
  2. gene correlation networks: data represent gene expression
- dense subsets of vertices correspond to highly correlated entities
- applications:
  1. analysis of stock market dynamics
  2. detecting co-expression modules

Motivation – fraud detection

- dense bipartite subgraphs in page-like data reveal attempts to inflate page-like counts [Beutel et al., 2013]



source: [Beutel et al., 2013]

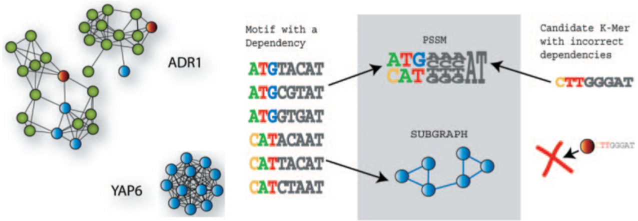
Motivation – e-commerce

e-commerce

- weighted bipartite graph $G(A \cup Q, E, w)$
- set A corresponds to advertisers
- set Q corresponds to queries
- each edge (a, q) has weight $w(a, q)$ equal to the amount of money advertiser a is willing to spend on query q

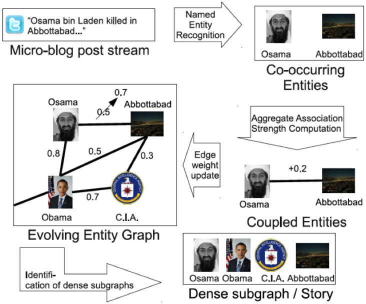
large almost bipartite cliques correspond to sub-markets

Motivation – bioinformatics



- DNA motif detection [Fratkin et al., 2006]
    - vertices correspond to k-mers
    - edges represent nucleotide similarities between k-mers
- gene correlation analysis
- detect complex annotation patterns from gene annotation data [Saha et al., 2010]

Motivation – mining twitter data



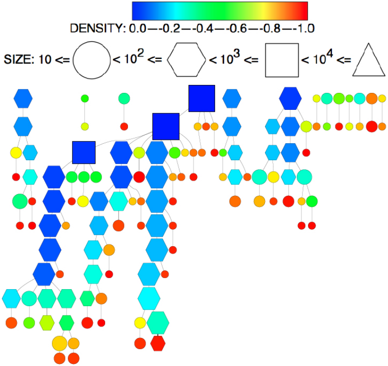
real-time story identification [Angel et al., 2012]

- mining of twitter data
- vertices correspond to entities
- edges correspond to co-occurence of entities
- dense subgraphs capture news stories

Motivation – graph mining

understanding the structure of real-world networks [Sarıyüce et al., 2015]

nucleus decomposition of a graph



(3,4)-nuclei forest for facebook

Motivation – distance queries in graphs

applications :

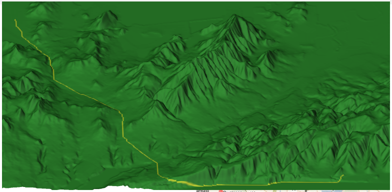
- driving directions
- indoor/terrain navigation
- routing in comm./sensor networks
- moving agents in game maps
- proximity in social/collab. networks

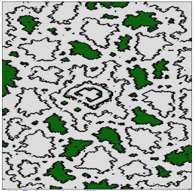
existing solutions :

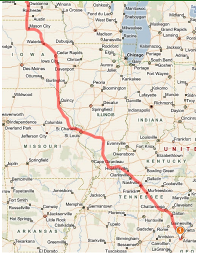
- graph searches are too slow
- fast algorithms are often heuristics
- or tailored to specific graph classes

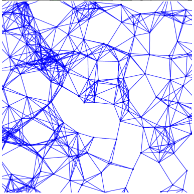
goals :

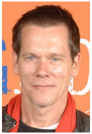
- fast exact queries
- scalability to large graphs
- wide range of inputs



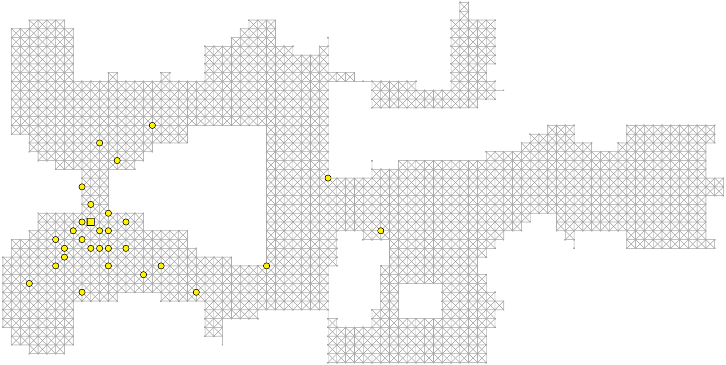








Motivation – distance queries in graphs

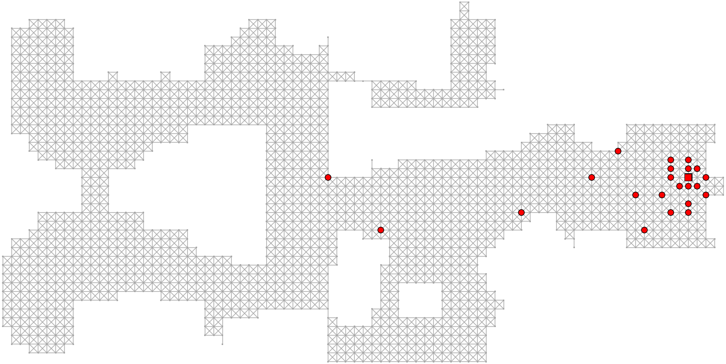


- $L(u) \equiv$ set of pairs $(v, \mathrm{dist}(u, v))$

  $L(u)$ is the *label* of u; each v is a *hub* for u.

figure from [Delling et al., 2014]

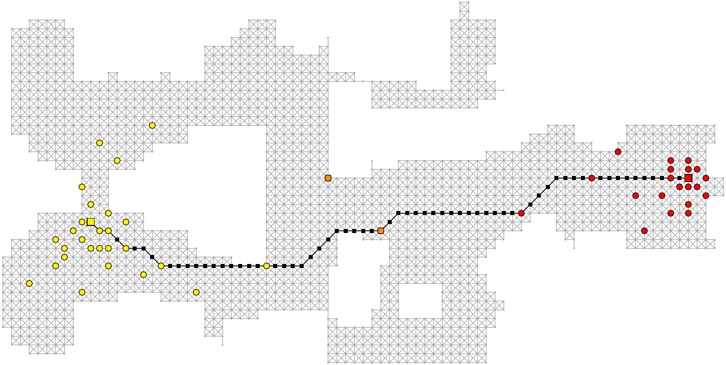
Motivation – distance queries in graphs



- preprocessing : compute a label set for every vertex
- cover property : for all s, t intersection $L(s) \cap L(t)$ must hit an s–t shortest path

figure from [Delling et al., 2014]

Motivation – distance queries in graphs



- to answer an s–t query :
  find hub v in $L(s) \cap L(t)$ minimizing $\text{dist}(s, v) + \text{dist}(v, t)$

figure from [Delling et al., 2014]

Motivation – distance queries in graphs

hub label queries are trivial to implement :

- entries sorted by hub id
- linear sweep to find matches
- access to only two contiguous blocks (cache-friendly)

method is practical if labels sets are small

- can we find small labels sets?
- 2-hop labeling algorithm relies on dense-subgraph discovery to find such label sets (!) [Cohen et al., 2003]
- state-of-art 2-hop labeling scheme : [Delling et al., 2014]
- more work on the topic : [Peleg, 2000, Thorup, 2004]

Motivation – frequent pattern mining

- given a set of transactions over items
- find item sets that occur together in a θ fraction of the transactions

issue number	heroes
1	Iceman, Storm, Wolverine
2	Aurora, Cyclops, Magneto, Storm
3	Beast, Cyclops, Iceman, Magneto
4	Cyclops, Iceman, Storm, Wolverine
5	Beast, Iceman, Magneto, Storm

e.g., {Iceman, Storm} appear in 60% of issues

Motivation – frequent pattern mining

- one of the most well-studied area in data mining
- many efficient algorithms

  Apriori, Eclat, FP-growth, Mafia, ABS, ...
- main idea: monotonicity

  a subset of a frequent set must be frequent, or

  a superset of an infrequent set must be infrequent
- algorithmically:

  start with small itemsets

  proceed with larger itemset if all subsets are frequent
- enumerate all frequent itemsets

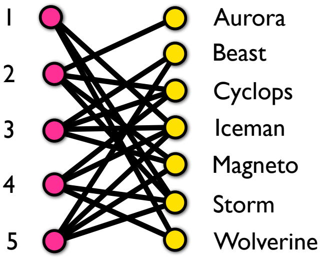
Motivation – frequent itemsets and dense subgraphs

id	heroes
1	Iceman, Storm, Wolverine
2	Aurora, Cyclops, Magneto, Storm
3	Beast, Cyclops, Iceman, Magneto
4	Cyclops, Iceman, Storm, Wolverine
5	Beast, Iceman, Magneto, Storm

⇔

	A	B	C	I	M	S	W
1	0	0	0	1	0	1	1
2	1	0	1	1	1	0	0
3	0	1	1	1	1	0	0
4	0	0	1	1	0	1	1
5	0	1	0	1	1	1	0

⇔



- transaction data ⇔ binary data ⇔ bipartite graphs

Motivation – frequent itemsets and dense subgraphs

id	heroes
1	Iceman, Storm, Wolverine
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⇔

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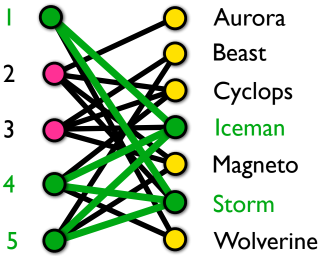
⇔



- transaction data ⇔ binary data ⇔ bipartite graphs
- frequent itemsets ⇔ bi-cliques

Motivation – finding web communities

[Kumar et al., 1999]

- hypothesis: web communities consist of hub-like pages and authority-like pages

  e.g., luxury cars and luxury-car aficionados

- key observations:

1. let $G = (U, V, E)$ be a dense web community

   then G should contain some small core (bi-clique)

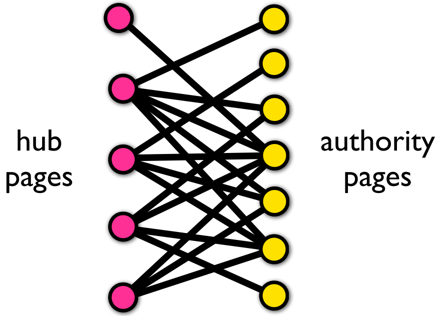
2. consider a web graph with no communities

   then small cores are unlikely

- both observations motivated from theory of random graphs

Motivation – finding web communities

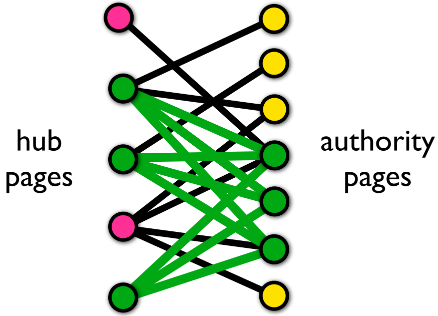
a web community



[Kumar et al., 1999]

Motivation – finding web communities

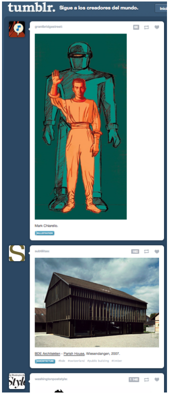
web communities containts small cores

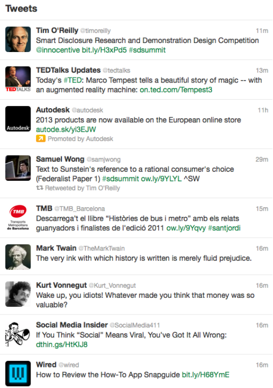


[Kumar et al., 1999]

Motivation – social piggybacking

[Gionis et al., 2013]

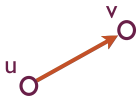




- event feeds: majority of activity in social networks

Motivation – social piggybacking

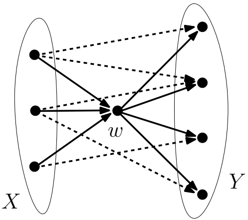
- system throughput proportional to the data transferred between data stores
- feed generation important component to optimize



- primitive operation: transfer data between two data stores
- can be implemented as push or pull strategy
- optimal strategy depends on production and consumption rates of nodes

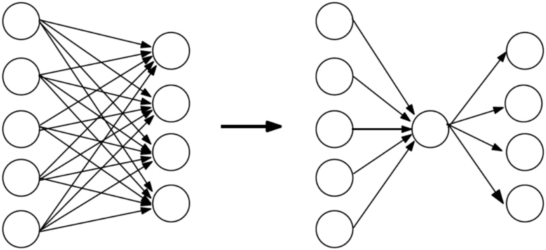
Motivation – social piggybacking

- hub optimization turns out to be a good idea
- depends on finding dense subgraphs



Motivation – graph compression

- compress web graphs by finding and compressing bi-cliques [Karande et al., 2009]
- many graph mining tasks that can be formulated as matrix-vector multiplication, are more efficient on the compressed graph [Kang et al., 2009]

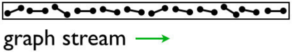


Motivation – more applications

- graph visualization [Alvarez-Hamelin et al., 2005]
- community detection [Chen and Saad, 2012]
- epilepsy prediction [Iasemidis et al., 2003]
- event detection in activity networks [Rozenshtein et al., 2014a]
- many more

Motivation – big and dynamic graphs

- size of graphs increases
  - e.g., in 2012, Facebook reported more than 1 billion users and 140 billion friend connections
- graphs change constantly
  - e.g., in Facebook friendships are created and deleted all the time
- need to design efficient algorithms on new computational models that handle large-scale processing
  - map-reduce, streaming models, etc.



Landscape of related work

- brute force [Johnson and Trick, 1996]
- heuristics [Bomze et al., 1999]
    - spectral algorithms [Alon et al., 1998, McSherry, 2001, Papailiopoulos et al., 2014]
    - belief-propagation methods [Kang et al., 2011]
- enumerating maximal cliques, e.g., [Bron and Kerbosch, 1973, Eppstein et al., 2010, Makino and Uno, 2004]
- NP-hard formulations and various relaxations
    - maximum clique problem [Karp, 1972, Hastad, 1999]
    - k-densest subgraph problem [Bhaskara et al., 2010, Feige et al., 2001]
    - optimal quasi-cliques [Tsourakakis et al., 2013]
- polynomial-time solvable objectives
    - densest subgraph problem [Goldberg, 1984]
    - *"The densest subgraph problem lies at the core of large scale data mining"* [Bahmani et al., 2012]

Preliminaries, measures of density

notation

- graph $G = (V, E)$ with vertices V and edges $E \subseteq V \times V$
- degree of a node $u \in V$ with respect to $X \subseteq V$ is

$$\deg_X(u) = |\{v \in X \text{ such that } (u, v) \in E\}|$$

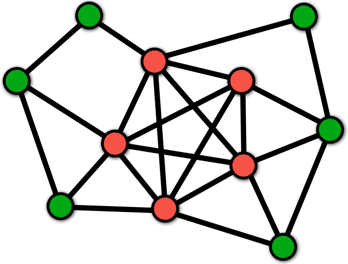
- degree of a node $u \in V$ is $\deg(u) = \deg_V(u)$
- edges between $S \subseteq V$ and $T \subseteq V$ are

$$E(S, T) = \{(u, v) \text{ such that } u \in S \text{ and } v \in T\}$$

  use shorthand $E(S)$ for $E(S, S)$
- graph cut is defined by a subset of vertices $S \subseteq V$
- edges of a graph cut $S \subseteq V$ are $E(S, \bar{S})$
- induced subgraph by $S \subseteq V$ is $G(S) = (S, E(S))$
- triangles: $T(S) = \{(u, v, w) \mid (u, v), (u, w), (v, w) \in E(S)\}$

density measures

- undirected graph $G = (V, E)$
- subgraph induced by $S \subseteq V$
- clique: all vertices in S are connected to each other



density measures

- edge density (average degree):

$$d(S) = \frac{2\,|E(S,S)|}{|S|} = \frac{2\,|E(S)|}{|S|}$$

(sometimes just drop 2)

- edge ratio:

$$\delta(S) = \frac{|E(S,S)|}{\binom{|S|}{2}} = \frac{|E(S)|}{\binom{|S|}{2}} = \frac{2\,|E(S)|}{|S|(|S|-1)}$$

- triangle density:

$$t(S) = \frac{|T(S)|}{|S|}$$

- triangle ratio:

$$\tau(S) = \frac{|T(S)|}{\binom{|S|}{3}}$$

other density measures

- *k*-core: every vertex in S is connected to at least k other vertices in S
- α-quasiclique: the set S has at least $\alpha \binom{|S|}{2}$ edges

  i.e., S is α-quasiclique if $E(S) \geq \alpha \binom{|S|}{2}$

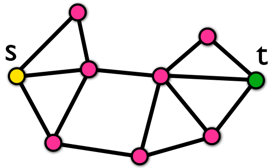
and more

not considered in this tutorial

- k-cliques: subset of vertices with pairwise distances at most k
  - distances defined using intermediaries, outside the set
  - not well connected
- k-club: a subgraph of diameter $\leq k$
- k-plex: a subgraph S in which each vertex is connected to at least $|S| - k$ other vertices
  - 1-plex is a clique

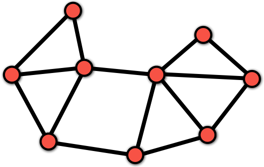
reminder: min-cut and max-cut problems

min-cut problem



- source $s \in V$, destination $t \in V$
- find $S \subseteq V$, s.t.,
- $s \in S$ and $t \in \bar{S}$, and
- minimize $e(S, \bar{S})$

max-cut problem



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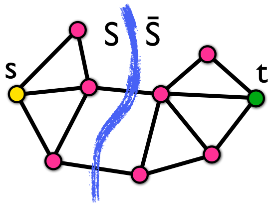
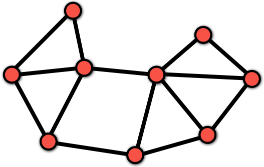
reminder: min-cut and max-cut problems

min-cut problem



- source $s \in V$, destination $t \in V$
- find $S \subseteq V$, s.t.,
- $s \in S$ and $t \in \bar{S}$, and
- minimize $e(S, \bar{S})$
- polynomially-time solvable
- equivalent to max-flow problem

max-cut problem



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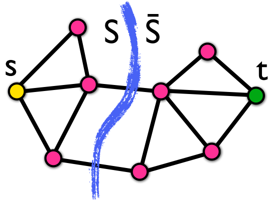
reminder: min-cut and max-cut problems

min-cut problem



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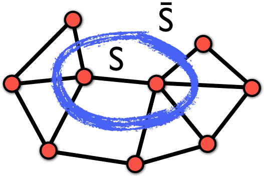
max-cut problem

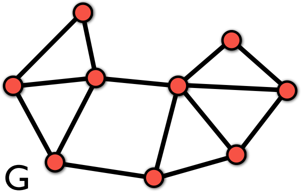


- find $S \subseteq V$, s.t.,
- maximize $e(S, \bar{S})$
- **NP**-hard
- approximation algorithms (0.868 based on SDP)

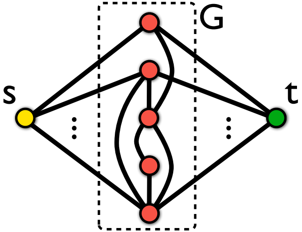
Efficient algorithms for static graphs

Goldberg's algorithm for densest subgraph

- consider first degree density d



- is there a subgraph S with $d(S) \geq c$?
- transform to a min-cut instance



- on the transformed instance:
- is there a cut smaller than a certain value?

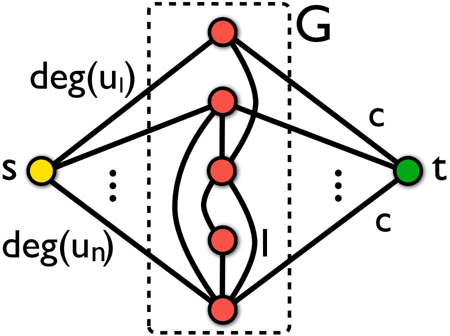
Goldberg's algorithm for densest subgraph

is there S with $d(S) \geq c$?

$$
\begin{aligned}
\frac{2\,|E(S,S)|}{|S|} &\geq c \\
2\,|E(S,S)| &\geq c|S| \\
\sum_{u\in S} \deg(u) - |E(S,\bar{S})| &\geq c|S| \\
\sum_{u\in S} \deg(u) + \sum_{u\in \bar{S}} \deg(u) - \sum_{u\in \bar{S}} \deg(u) - |E(S,\bar{S})| &\geq c|S| \\
\sum_{u\in \bar{S}} \deg(u) + |E(S,\bar{S})| + c|S| &\leq 2\,|E|
\end{aligned}
$$

Goldberg's algorithm for densest subgraph

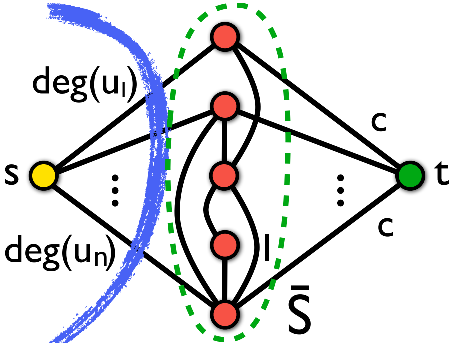
- transformation to min-cut instance



- is there S s.t. $\sum_{u \in \bar{S}} \deg(u) + |e(S, \bar{S})| + c|S| \leq 2\,|E|$?

Goldberg's algorithm for densest subgraph

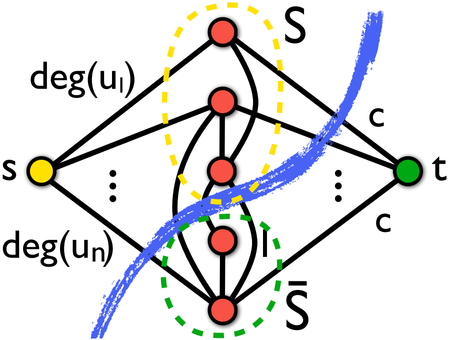
- transform to a min-cut instance



- is there S s.t. $\sum_{u \in \bar{S}} \deg(u) + |e(S, \bar{S})| + c|S| \leq 2\,|E|$?
- a cut of value $2\,|E|$ always exists, for $S = \emptyset$

Goldberg's algorithm for densest subgraph

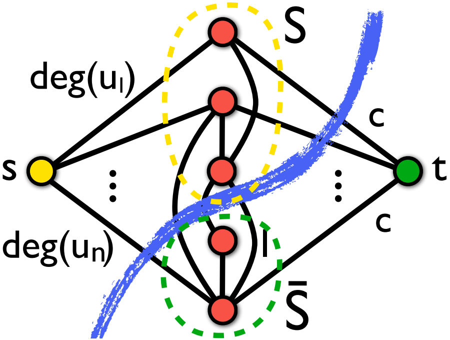
- transform to a min-cut instance



- is there S s.t. $\sum_{u \in \bar{S}} \deg(u) + |e(S, \bar{S})| + c|S| \leq 2\,|E|$?
- $S \neq \emptyset$ gives cut of value $\sum_{u \in \bar{S}} \deg(u) + |e(S, \bar{S})| + c|S|$

Goldberg's algorithm for densest subgraph

- transform to a min-cut instance



- is there S s.t. $\sum_{u\in\bar{S}} \deg(u) + |e(S,\bar{S})| + c|S| \leq 2\,|E|$?
- YES, if min cut achieved for $S \neq \emptyset$

Goldberg's algorithm for densest subgraph

[Goldberg, 1984]

input: undirected graph $G = (V, E)$, number c
output: S, if $d(S) \geq c$
transform G into min-cut instance $G' = (V \cup \{s\} \cup \{t\}, E', w')$
find min cut $\{s\} \cup S$ on G'
if $S \neq \emptyset$ return S
else return NO

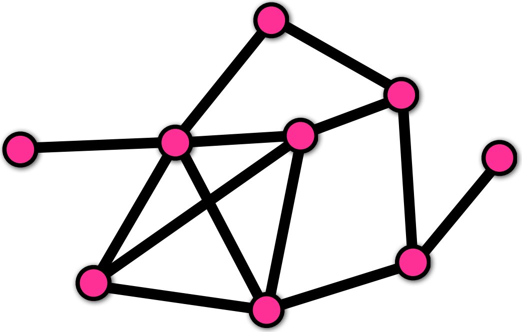
- to find the densest subgraph perform binary search on c
- logarithmic number of min-cut calls
- problem can also be solved with one min-cut call using the parametric max-flow algorithm

densest subgraph problem – discussion

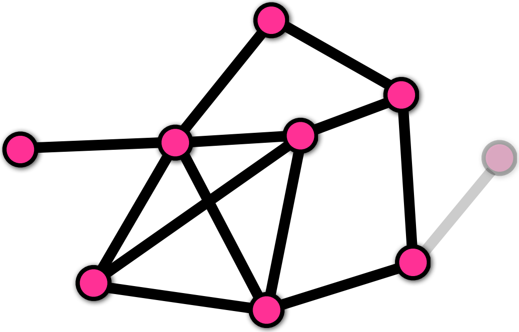
- Goldberg's algorithm polynomial algorithm, but
- $\mathcal{O}(nm)$ time for one min-cut computation
- not scalable for large graphs (millions of vertices / edges)

- faster algorithm due to [Charikar, 2000]
- greedy and simple to implement
- approximation algorithm

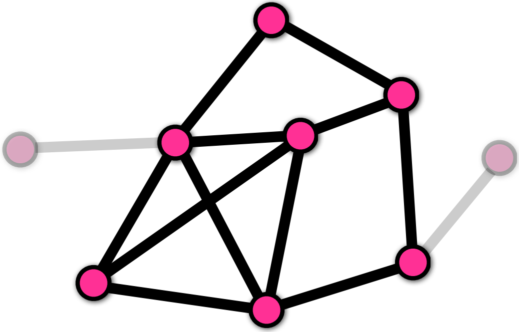
greedy algorithm for densest subgraph — example



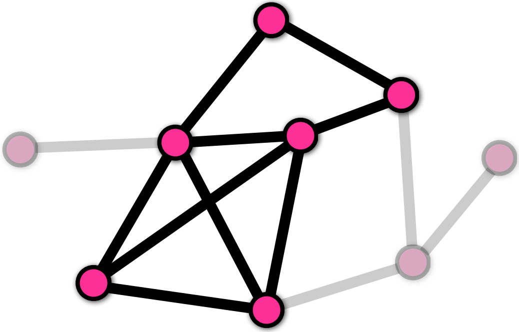
greedy algorithm for densest subgraph — example



greedy algorithm for densest subgraph — example



greedy algorithm for densest subgraph — example



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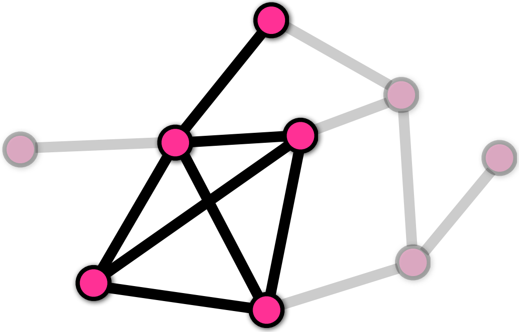

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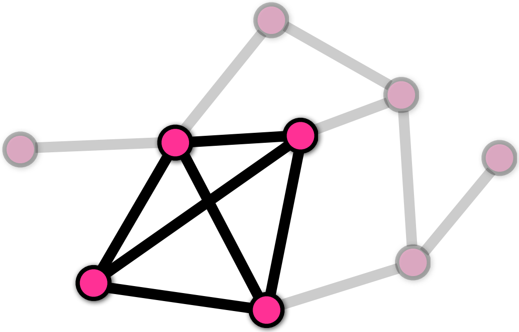

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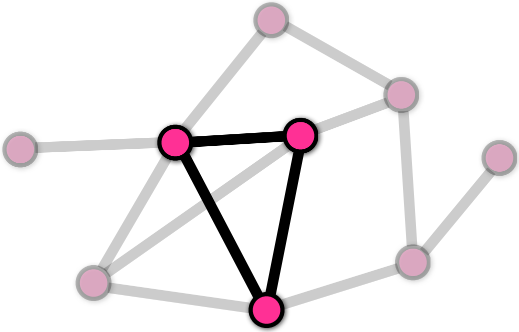
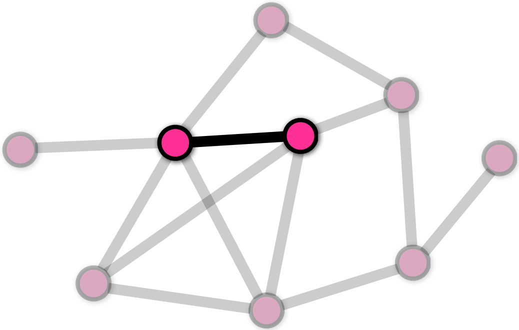
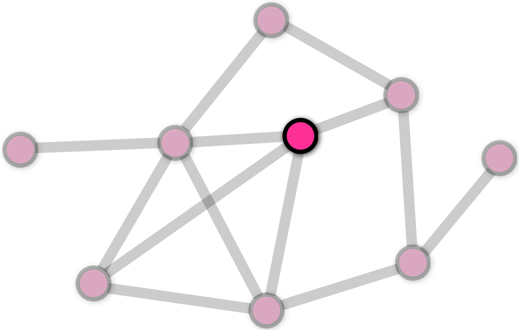


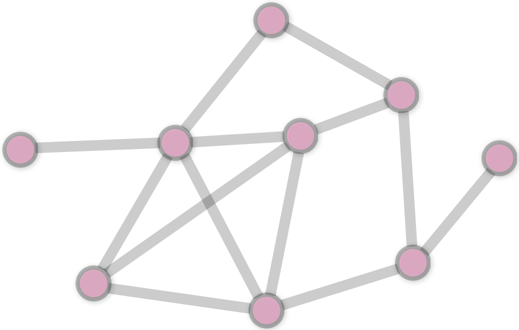
greedy algorithm for densest subgraph — example



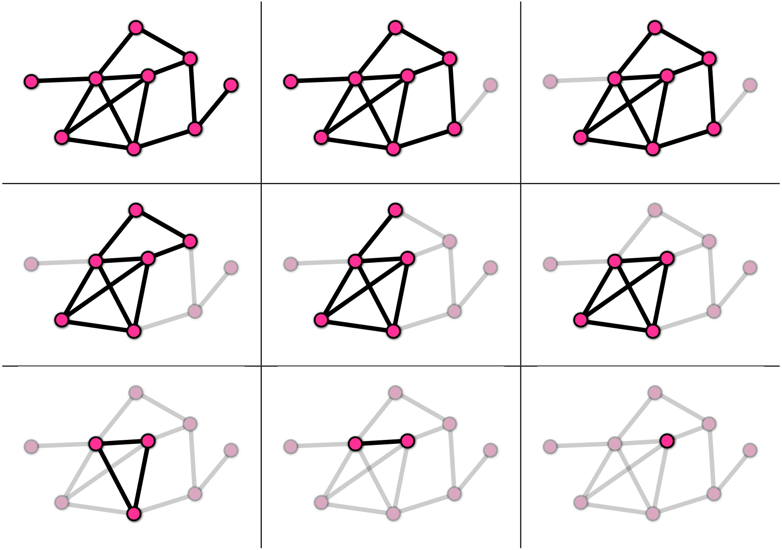
greedy algorithm for densest subgraph — example



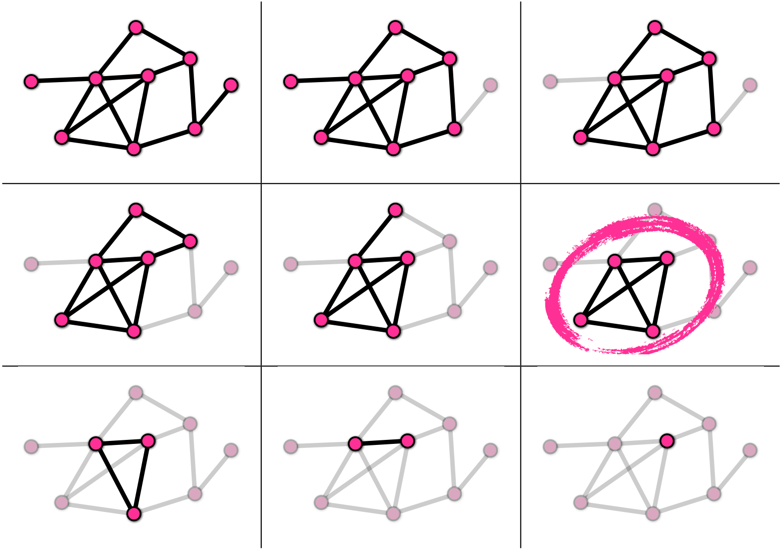
greedy algorithm for densest subgraph — example



greedy algorithm for densest subgraph — example



greedy algorithm for densest subgraph — example



greedy algorithm for densest subgraph

[Charikar, 2000]

input: undirected graph $G = (V, E)$
output: S, a dense subgraph of G

1 set $G_n \leftarrow G$
2 for $k \leftarrow n$ downto 1
2.1 let v be the smallest degree vertex in G_k
2.2 $G_{k-1} \leftarrow G_k \setminus \{v\}$
3 output the densest subgraph among $G_n, G_{n-1}, \ldots, G_1$

proof of 2-approximation guarantee

a neat argument due to [Khuller and Saha, 2009]

- let S^* be the vertices of the optimal subgraph
- let $d(S^*) = \lambda$ be the maximum degree density
- notice that for all $v \in S^*$ we have $\deg_{S^*}(v) \geq \lambda$
- (why?) by optimality of S^*

$$\frac{|e(S^*)|}{|S^*|} \geq \frac{|e(S^*)| - \deg_{S^*}(v)}{|S^*| - 1}$$

and thus

$$\deg_{S^*}(v) \geq \frac{|e(S^*)|}{|S^*|} = d(S^*) = \lambda$$

proof of 2-approximation guarantee (continued)

([Khuller and Saha, 2009])

- consider greedy when the first vertex $v \in S^* \subseteq V$ is removed
- let S be the set of vertices, just before removing v
- total number of edges before removing v is $\geq \lambda|S|/2$
- therefore, greedy returns a solution with degree density at least $\frac{\lambda}{2}$

QED

the greedy algorithm

- factor-2 approximation algorithm
- runs in linear time $\mathcal{O}(n + m)$
- for a polynomial problem ...

  but faster and easier to implement than the exact algorithm

- everything goes through for weighted graphs

  using heaps: $\mathcal{O}(m + n \log n)$

- things are not as straightforward for directed graphs

Dense subgraphs on directed graphs – history

- goal: find sets $S, T \subseteq V$ to maximize

$$d(S,T) = \frac{e[S,T]}{\sqrt{|S|\,|T|}}$$

- first introduced in unpublished manuscript [Kannan and Vinay, 1999]
- they provided a $\mathcal{O}(\log n)$-approximation algorithm
- left open the problem complexity

- polynomial-time solution using linear programming (LP) [Charikar, 2000]

Dense subgraphs on directed graphs – history

[Charikar, 2000]

- exact LP-based algorithm
- greedy 2-approximation algorithm running in $\mathcal{O}(n^3 + n^2 m)$

[Khuller and Saha, 2009]

- first max-flow based exact algorithm
- improved running time of the 2-approximation greedy algorithm to $\mathcal{O}(n + m)$!

Directed graphs – algorithms

- reduced problem to $O(n^2)$ LP calls [Charikar, 2000]
- one LP call for each possible ratio $\frac{|S|}{|T|} = c$

$$
\begin{array}{lll}
\text{maximize} & \displaystyle\sum_{(i,j)\in E(G)} x_{ij} & \\
\text{such that} & x_{ij} \le s_i, & \text{for all } (i,j) \in E(G) \\
 & x_{ij} \le t_j, & \text{for all } (i,j) \in E(G) \\
 & \displaystyle\sum_i s_i \le \sqrt{c} \text{ and } \sum_j t_j \le \frac{1}{\sqrt{c}} & \\
 & x_{ij}, s_i, t_j \ge 0 &
\end{array}
$$

Dense subgraphs on directed graphs – greedy

[Charikar, 2000]

input: directed graph $G = (V, E)$, ratio $c = \frac{|S|}{|T|}$

$S \leftarrow V$, $T \leftarrow V$
while both S, T non-empty
  $i_{\min} \leftarrow$ the vertex $i \in S$ that minimizes $|E(\{i\}, T)|$
  $d_S \leftarrow |E(\{i_{\min}\}, T)|$
  $j_{\min} \leftarrow$ the vertex $j \in T$ that minimizes $|E(S, \{j\})|$
  $d_T \leftarrow |E(S, \{j_{min}\})|$
  if $\sqrt{c} d_S \leq \frac{1}{\sqrt{c}} d_T$
    then $S \leftarrow S \backslash \{i_{\min}\}$
    else $ST \leftarrow T \backslash \{j_{\min}\}$

- execute $\mathcal{O}(n^2)$ times; one for each $c = \frac{|S|}{|T|}$
- report best solution
- factor 2 approximation guarantee

Dense subgraphs on directed graphs – greedy

- brute force execution of greedy: $\mathcal{O}(n^2(n+m)) = \mathcal{O}(n^3 + nm))$

[Khuller and Saha, 2009]

- showed that only one execution is needed (instead of $\mathcal{O}(n^2)$)
- total running time $\mathcal{O}(n+m)$

Dense subgraphs on directed graphs – greedy

linear-time greedy [Khuller and Saha, 2009]

definitions:

- let v_i, v_o be the vertices with minimum in- and out-degree
- if $d^-(v_i) \leq d^+(v_o)$ we are in category IN
  otherwise in category OUT

algorithm:

- greedy deletes the minimum-degree vertex
- if in IN, it deletes all incoming edges
- if in OUT, it deletes all outgoing edges
- if the vertex becomes a singleton, it is deleted.

- return the densest subgraph encountered

Dense subgraphs on directed graphs – exact

we wish to answer "are there $S, T \subseteq V$ such that $d(S,T) \geq g$?"

consider

- consider $\alpha = \frac{|S|}{|T|}$ ($\mathcal{O}(n^2)$ possible values)
- network $G' = (\{s,t\} \cup V_1 \cup V_2, E)$, with $V_1 = V_2 = V$

min-cut transformation

- add an edge of capacity m from s to each vertex of V_1 and V_2
- add an edge of capacity $2m + \frac{g}{\sqrt{\alpha}}$ from each vertex of V_1 to t
- add an edge from each vertex j of V_2 to sink t of capacity $2m + \sqrt{\alpha}g - 2\mathrm{deg}(j)$
- for each $(i,j) \in E(G)$, add an edge from $j \in V_2$ to $i \in V_1$ with capacity 2

Dense subgraph problem – summary

- for the degree density measure:
- exact algorithms for undirected and directed graphs
- linear-time 2-approximation achieved by greedy

- how good are these subgraphs?
  study other measures and contrast with degree density

- no control on the size of the subgraph

- what about time-evolving and dynamic graphs?

Edge-surplus framework

introduced by [Tsourakakis et al., 2013]

- for a set of vertices S define edge surplus

$$f(S) = g(e[S]) - h(|S|)$$

where g and h are both strictly increasing

- optimal (g, h)-edge-surplus problem:

  find S^* such that

$$f(S^*) \geq f(S), \quad \text{for all sets} \quad S \subseteq S^*$$

Edge-surplus framework

- edge surplus $f(S) = g(e[S]) - h(|S|)$

- example 1
$$g(x) = h(x) = \log x$$
find S that maximizes $\log \frac{e[S]}{|S|}$
densest-subgraph problem

- example 2
$$g(x) = x, \quad h(x) = \begin{cases} 0 & \text{if } x = k \\ +\infty & \text{otherwise} \end{cases}$$
k-densest-subgraph problem

The optimal quasiclique problem

- edge surplus $f(S) = g(e[S]) - h(|S|)$
- consider

$$g(x) = x, \quad h(x) = \alpha \frac{x(x-1)}{2}$$

  find S that maximizes $e[S] - \alpha \binom{|S|}{2}$

  optimal quasiclique problem [Tsourakakis et al., 2013]

- theorem: let $g(x) = x$ and $h(x) = \alpha x$
  we aim to maximize $e(S) - \alpha|S|$
  solving $\mathcal{O}(\log n)$ such problems, solves densest subgraph problem

The edge-surplus maximization problem

theorem: let $g(x) = x$ and $h(x)$ concave

then the optimal (g, h)-edge-surplus problem is polynomially-time solvable

proof

$g(x) = x$ is supermodular

if $h(x)$ concave $h(x)$ is submodular

$-h(x)$ is supermodular

$g(x) - h(x)$ is supermodular

maximizing supermodular functions is a polynomial problem

The edge-surplus maximization problem

- poly-time solvable and interesting objectives have linear h
- the optimal quasiclique problem is NP-hard [Tsourakakis, 2014]
- the partitioning version led to a state-of-art streaming balanced graph-partitioning algorithm: FENNEL
  - goal: maximize $g(\mathcal{P})$ over all possible k-partitions
  - notice:

$$g(\mathcal{P}) = \underbrace{\sum_i e[S_1]}_{\text{number of edges cut}} - \alpha \underbrace{\sum_i |S_i|^\gamma}_{\text{minimized for balanced partition!}}$$

  - for more details: [Tsourakakis et al., 2014]

Finding optimal quasicliques

adaptation of the greedy algorithm of [Charikar, 2000]

input: undirected graph $G = (V, E)$
output: a quasiclique S
1 set $G_n \leftarrow G$
2 for $k \leftarrow n$ downto 1
2.1 let v be the smallest degree vertex in G_k
2.2 $G_{k-1} \leftarrow G_k \setminus \{v\}$
3 output the subgraph in $G_n, \ldots, G_1$ that maximizes $f(S)$

additive approximation guarantee [Tsourakakis et al., 2013]

Motivating research question

- despite rich landscape of algorithmic tools, until recently, no polynomial algorithm for finding large near-cliques

- can we combine the best of both worlds, namely
  - have poly-time solvable formulation(s) which ...
  - ...consistently succeeds in finding large near-cliques on real-world networks?

- yes! the k-clique densest subgraph problem [Tsourakakis, 2015]

k-clique densest subgraph problem

Definition (k-clique density)

For any $S \subseteq V$ we define its k-clique density $\rho_k(S)$, $k \geq 2$ as $\rho_k(S) = \frac{c_k(S)}{s}$, where $c_k(S)$ is the number of k-cliques induced by S and $s = |S|$

Problem (k-clique DSP)

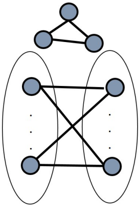
Given $G(V, E)$, find a subset of vertices S^* such that $\rho_k(S^*) = \rho_k^* = \max_{S \subseteq V} \rho_k(S)$

- Notice that the 2-clique DSP is simply the DSP
- We shall refer to the 3-clique DSP as the *triangle densest subgraph problem*

$$\max_{S \subseteq V} \tau(S) = \frac{t(S)}{s}$$

Triangle densest subgraph problem

- How different can the densest subgraph be from the triangle densest subgraph?
  In principle, they can be radically different!
  Consider $G = K_{n,n} \cup K_3$



- The interesting question is what happens on real-data
- Can we solve the triangle DSP in polynomial time?
- Can we solve the k-clique DSP in polynomial time?

Triangle densest subgraph problem

Theorem

There exists an algorithm which solves the TDSP and runs in $O\left(m^{3/2} + nt + \min(n,t)^3\right)$ *time*

We will sketch here the idea behind a $O\left(m^{3/2} + \left(nt + \min(n,t)^3\right)\log n\right)$ algorithm Furthermore,

Theorem

We can solve the k-clique DSP in polynomial time for any $k = \Theta(1)$

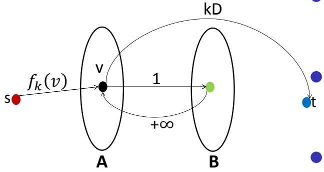
- Even if our construction solves the DSP, Goldberg's algorithm is more efficient

Triangle densest subgraph problem

- Perform binary searches:
  - $\exists S \subseteq V$ such that $t(S) > \alpha|S|$?
- $\mathcal{O}(\log n)$ queries suffice in order to solve the TDSP
  - Any two distinct triangle density values are at least $\mathcal{O}(1/n^2)$ way from each other
  - The optimal density $0 \le \frac{t}{n} \le \tau^* \le \frac{\binom{n}{3}}{n}$
- But what does a binary search correspond to? ...

Triangle densest subgraph problem

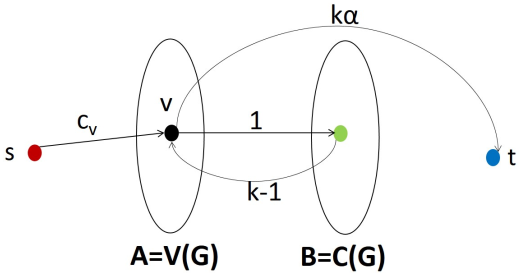
. . . To a maximum flow computation on this network



Construct-Network $(G, \alpha, \mathcal{T}(G))$

- $V(H) \leftarrow \{s\} \cup V(G) \cup \mathcal{T}(G) \cup \{t\}$
- For each vertex $v \in V(G)$ add an arc of capacity 1 to each triangle t_i it participates in
- For each triangle $\Delta = (u, v, w) \in \mathcal{T}(G)$ add arcs to u, v, w of capacity 2
- Add directed arc $(s, v) \in A(H)$ of capacity t_v for each $v \in V(G)$
- Add weighted directed arc $(v, t) \in A(H)$ of capacity 3α for each $v \in V(G)$
- Return network $H(V(H), A(H), w), s, t \in V(H)$

k-clique densest subgraph problem



Triangle densest subgraph problem

Exact-TDS

- List the set of triangles $\mathcal{T}(G)$, $t = |\mathcal{T}(G)|$
- $l \leftarrow \frac{t}{n}$, $u \leftarrow \frac{(n-1)(n-2)}{6}$
- $S^* \leftarrow \emptyset$
- While$(u \geq l + \frac{1}{n(n-1)})$
  - $\alpha \leftarrow \frac{l+u}{2}$
  - $H_\alpha \leftarrow$ Construct-Network$(G, \alpha, \mathcal{T}(G))$
  - $(S, T) \leftarrow$ minimum st-cut in H_α
  - If($S = \{s\}$), then $u \leftarrow \alpha$
  - otherwise set $S^* \leftarrow \big(S \backslash \{s\}\big) \cap V(G)$ and $l \leftarrow \alpha$
- Return S^*

1. Run time: $O\Big(m^{3/2} + \big(nt + \min{(n,t)}^3\big) \log n\Big)$
2. Space complexity: $\mathcal{O}(n+t)$. Typically $n \ll t$ on real networks

Triangle densest subgraph problem

1. Set $G_n \leftarrow G$
2. for $k \leftarrow n$ downto 1
   - Let v be the smallest triangle count vertex in G_k
   - $G_{k-1} \leftarrow G_k \backslash \{v\}$
3. Output the triangle densest subgraph among $G_n, G_{n-1}, \ldots, G_1$

- The above peeling algorithm is a 3-approximation algorithm

- The same peeling idea generalizes to the k-clique DSP, providing a k-approximation algorithm

Some experimental findings

Method	Measure	Football
DS	$\frac{\lvert S\rvert}{\lvert V\rvert}(\%)$	100
	2δ	10.6
	f_e	0.094
	3τ	21.12
$\frac{1}{2}$-DS	$\frac{\lvert S\rvert}{\lvert V\rvert}(\%)$	100
	2δ	10.66
	f_e	0.094
	3τ	21.12

Method	Measure	Football
TDS	$\frac{\lvert S\rvert}{\lvert V\rvert}(\%)$	15.7
	2δ	8.22
	f_e	0.48
	3τ	28
$\frac{1}{3}$-TDS	$\frac{\lvert S\rvert}{\lvert V\rvert}(\%)$	15.7
	2δ	8.22
	f_e	0.48
	3τ	28

- Observation 1. Approximate counterparts are close to the optimal exact methods
- Observation 2. The TDS is closer to being a large near-clique compared to the DS

Important remark

- Charikar's algorithm despite being a 2-approximation algorithm performs optimally or close to optimally on real data. This suggests that real-data are "far away" from being adversarial

- Here is one adversarial instance that shows that the 2-approximation is tight

– $G = G_1 \cup G_2$ where $G_1 = K_{d,D}$, G_2 is the disjoint union of D cliques, each of size $d + 1$

– Let $d \ll D$

– How does the Charikar's algorithm perform?

  – Instead of returning the bipartite clique with density $dD/(d + D) \approx d$, it returns a clique of size $d + 1$ with density $d/2$

Computational issues

- The main issue is the size of the bipartite network
  - Both space-wise ...
  - and time-wise, as any max-flow computation depends on its size
- *k*-clique counting is not the main issue. We can count fast based on arboricity based ordering heuristics *k*-cliques efficiently on large networks
  - When the counting part becomes an issue, high-quality approximation algorithms exist, e.g., [Kolountzakis et al., 2012, Tsourakakis et al., 2011, Pagh and Tsourakakis, 2012]

Datasets

Name	n	m
■ Web-Google	875 713	3 852 985
⋆ Epinions	75 877	405 739
⊙ CA-Astro	18 772	198 050
■ Pol-blogs	1 222	16 714
⊙ Email-all	234 352	383 111
■ LastFm-B	17 644	92 366
⋆ IMDB-B	241 360	530 494
⋆ IMDB-G-B	21 258	42 197
⊙ Bookmarks-B	71 090	437 593

Experimental findings

k-cliques

G	$k = 2$		$k = 3$		$k = 4$		$k = 5$	
	f_e	$\|S\|$	f_e	$\|S\|$	f_e	$\|S\|$	f_e	$\|S\|$
$\star$	0.12	1 012	0.26	432	0.40	235	0.50	172
$\odot$	0.11	18 686	0.80	76	0.96	62	0.96	62
$\blacksquare$	0.19	16 714	0.54	102	0.59	92	0.63	84
$\odot$	0.13	553	0.38	167	0.48	122	0.53	104

(p,q)-bicliques

G	$(p, q) = (1, 1)$		$(p, q) = (2, 2)$		$(p, q) = (3, 3)$	
	f_e	$\|S\|$	f_e	$\|S\|$	f_e	$\|S\|$
$\star$	0.001	9 177	0.06	181	0.30	40
$\star$	0.001	6 437	0.41	18	0.43	17

Densest subgraph sparsifiers

Abstraction: We shall abstract both the k-clique DSP and the (p, q)-biclique DSP as a densest subgraph problem in a hypergraph. Let $\mathcal{H}$ be the resulting hypergraph and $\epsilon > 0$ be an accuracy parameter

Theorem

- *Sample each hyperedge $e \in E_{\mathcal{H}}$ independently with probability $p = \frac{6}{\epsilon^2}\frac{\log n}{D}$*
- *Then, the following statements hold simultaneously with high probability:*
  - *For all $U \subseteq V$ such that $\rho(U) \geq D$, $\tilde{\rho}(U) \geq (1-\epsilon)C \log n$ for any $\epsilon > 0$*
  - *For all $U \subseteq V$ such that $\rho(U) < (1-2\epsilon)D$, $\tilde{\rho}(U) < (1-\epsilon)C \log n$ for any $\epsilon > 0$*

Densest subgraph sparsifiers

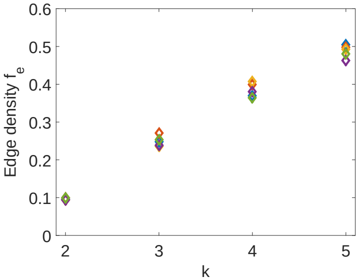
Technical difficulty

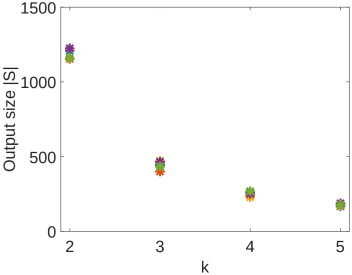
- Notice that taking Chernoff bounds and a union bound does not work since by Chernoff the failure probability is $1/poly(n)$ whereas there exists an exponential number of potential bad events

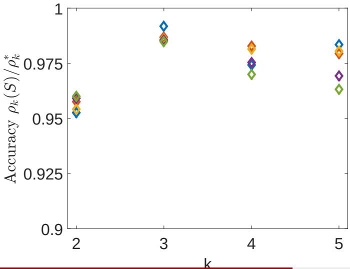
From the previous theorem, we obtain the following corollaries

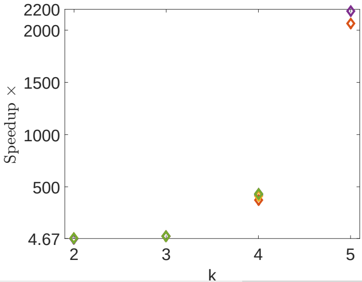
- $(1+\Theta(\epsilon))$-approximation, expected speedup $\mathcal{O}(\frac{1}{p_D^2})$, expected space reduction is $\mathcal{O}(\frac{1}{p_D})$
- Naturally results in a single pass $(1+\Theta(\epsilon))$-approximation semi-streaming algorithm for a dynamic stream of edges. Same result obtained independently by [Esfandiari et al., 2015, McGregor et al., 2015]

Sampling effect, Epinions network









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Densest subgraph sparsifiers

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- Notice that taking Chernoff bounds and a union bound does not work since by Chernoff the failure probability is $1/poly(n)$ whereas there exists an exponential number of potential bad events

From the previous theorem, we obtain the following corollaries

- $(1+\Theta(\epsilon))$-approximation, expected speedup $\mathcal{O}(\frac{1}{p_D^2})$, expected space reduction is $\mathcal{O}(\frac{1}{p_D})$
- We can sample with probability $p = \Theta\left(\frac{n \log n}{\epsilon^2 m_{\mathcal{H}}}\right)$, where $m_{\mathcal{H}} = |E_{\mathcal{H}}|$
- Our sampling scheme results in a single pass $(1+\Theta(\epsilon))$-approximation semi-streaming algorithm for DSP. Same result obtained later independently by [Esfandiari et al., 2015, McGregor et al., 2015]

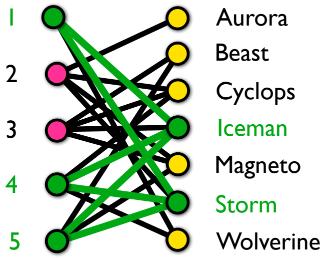
Large Near Bicliques

id	heroes
1	Iceman, Storm, Wolverine
2	Aurora, Cyclops, Magneto, Storm
3	Beast, Cyclops, Iceman, Magneto
4	Cyclops, Iceman, Storm, Wolverine
5	Beast, Iceman, Magneto, Storm

$\Leftrightarrow$

	A	B	C	I	M	S	W
1	0	0	0	1	0	1	1
2	1	0	1	1	1	0	0
3	0	1	1	1	1	0	0
4	0	0	1	1	0	1	1
5	0	1	0	1	1	1	0

$\Leftrightarrow$



- transaction data $\Leftrightarrow$ binary data $\Leftrightarrow$ bipartite graphs
- frequent itemsets $\Leftrightarrow$ bi-cliques

Large Near Bicliques

- We generalize the idea of k-cliques by maximizing the average (p, q)-biclique densities
- For $p = q = 1$ we obtain the well-known densest subgraph problem
- We provide general network construction techniques which can be used to maximize the (p, q)-biclique density for any $p, q = \Theta(1)$
  - Our network construction techniques can be used to maximize densities of other types of subgraphs as well
- We can justify speedups of the order $\mathcal{O}(\rho^{*2}/\log^2 n)$, compared to the exact maximum flow computation based algorithm

Datasets

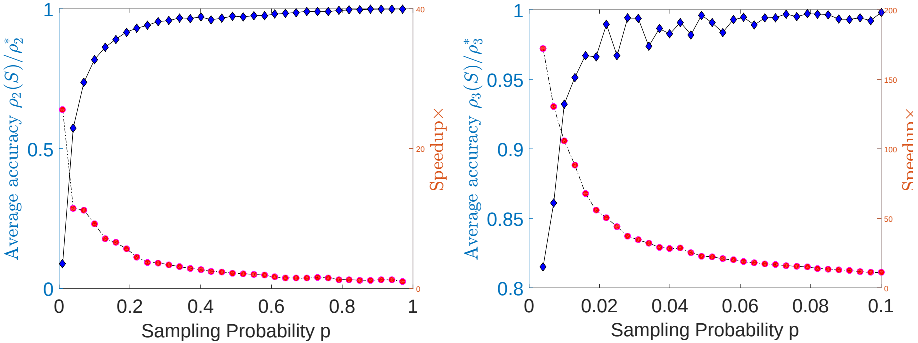
Name	n	m
■ Web-Google	875 713	3 852 985
⋆ Epinions	75 877	405 739
⊙ CA-Astro	18 772	198 050
■ Pol-blogs	1 222	16 714
⊙ Email-all	234 352	383 111
■ LastFm-B	17 644	92 366
⋆ IMDB-B	241 360	530 494
⋆ IMDB-G-B	21 258	42 197
⊙ Bookmarks-B	71 090	437 593

k-clique and (p, q)-biclique counts and run times

Name	c_3	T	c_4	T	c_5	T
■ Web-Google	11.4M	8.5	32.5M	16.5	82M	36.4
⋆ Epinions	1 6M	1.6	5.8M	4.8	17.5M	13.4
⊙ CA-Astro	1 3M	0.6	9.6M	3.94	65M	27.2
■ Pol-blogs	101K	0.05	422K	0.2	1.4M	0.7
⊙ Email-all	383K	0.4	1.1M	0.9	2.7M	1.9

Name	$c_{2,2}$	T	$c_{3,3}$	T
■ LastFm-B	18 266 703	27.8	-	-
⋆ IMDB-B	691 594	3.6	261 330	3.3
⋆ IMDB-G-B	14 919	0.1	2 288	0.1
⊙ Bookmarks-B	431 996	0.82	14 901	0.53

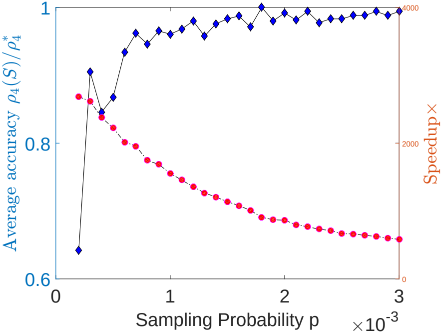
Ranging p, $k = 2, 3$



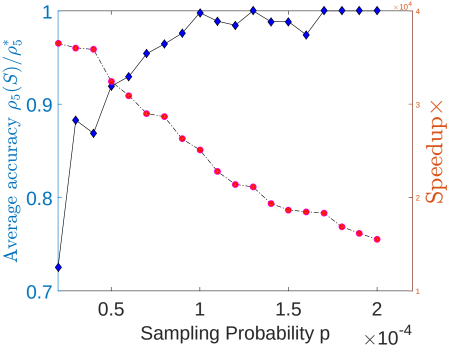
k=2 k=3

Accuracy $\rho_k(S)/\rho_k^*$ and speedup as functions of the sampling probability p for the CA-Astro collaboration network

Ranging p, $k = 4, 5$



k=4



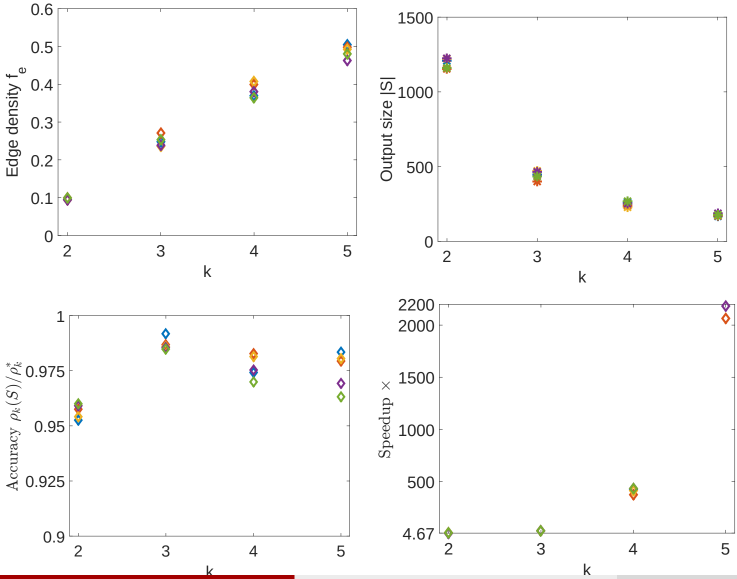
k=5

Accuracy $\rho_k(S)/\rho_k^*$ and speedup as functions of the sampling probability p for the CA-Astro collaboration network

Observations – Ranging p

- Notice that $\frac{c_k}{n} \leq \rho_k^* \leq \frac{\binom{n}{k}}{n}$
- We observe that an efficient strategy is to guess a large value of ρ_k^*, i.e., sample with smallest value for p Then, while concentration is not deduced, keep doubling p
- The speedups for $k = 2$ -while valuable- are not impressive as the graphs are pretty sparse to begin with
- However, for $k \geq 3$ the speedups start becoming significant, reaching the order of 4×10^4 for $k = 5$, which achieving excellent accuracies

Sampling effect, Epinions



Accuracies and speedups

- Runtimes (exact), accuracies and speedups (random sampling)
  - Exact: For $k = 2$ the slowest run time was 33.9 secs
  - Sampling: We obtain a speedup of $\approx 3\times$ using sampling
    Accuracies greater always than 95%
  - Exact: For $k = 5$, the exact algorithm cannot run on one dataset
    Run times for other datasets, 37 939.6, 2 107.2, 24.04, 52.4
  - Sampling: Speedups range from $410.3\times$ to $77\,288\times$. Accuracies close to 100%
- The results for $k = 3, 4$ interpolate. For the detailed findings, please look at our paper

Effect of hierarchy

k-cliques

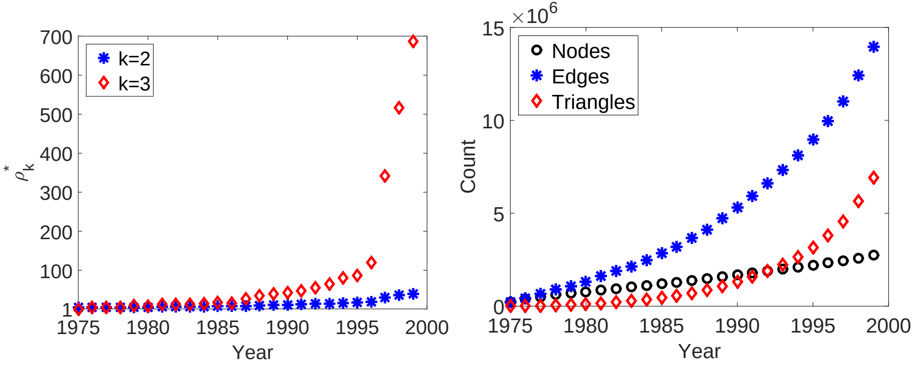
G	$k=2$		$k=3$		$k=4$		$k=5$	
	f_e	$\lvert S\rvert$	f_e	$\lvert S\rvert$	f_e	$\lvert S\rvert$	f_e	$\lvert S\rvert$
$\star$	0.12	1 012	0.26	432	0.40	235	0.50	172
$\odot$	0.11	18 686	0.80	76	0.96	62	0.96	62
■	0.19	16 714	0.54	102	0.59	92	0.63	84
$\odot$	0.13	553	0.38	167	0.48	122	0.53	104

(p,q)-bicliques

G	$(p,q)=(1,1)$		$(p,q)=(2,2)$		$(p,q)=(3,3)$	
	f_e	$\lvert S\rvert$	f_e	$\lvert S\rvert$	f_e	$\lvert S\rvert$
$\star$	0.001	9 177	0.06	181	0.30	40
$\star$	0.001	6 437	0.41	18	0.43	17

Time evolving networks

Patents citation network that spans 37 years, specifically from January 1, 1963 to December 30, 1999.

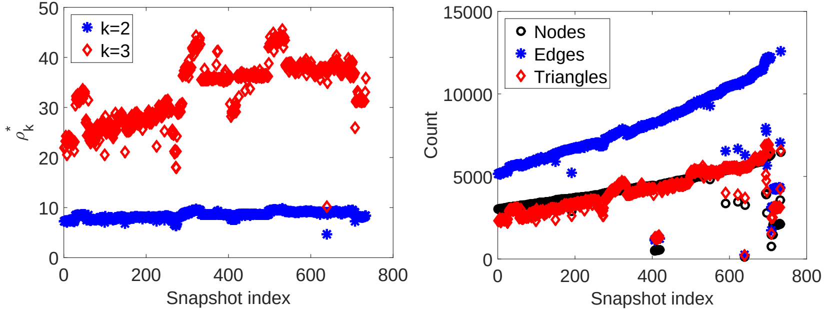


Time evolving networks

- We observe in the left Figure that both ρ_2^* and ρ_3^* exhibit an increasing trend.
- This increasing trend becomes is mild for ρ_3^* up to 1995, but then it takes off
- What makes this finding even more interesting as the number of edges grows faster than the number of triangles
- We are seeing an outlier - the company Allergan, Inc. This company tends to cite all their previous patents with each new patent and creates a dense subregion in the graph

Time evolving networks

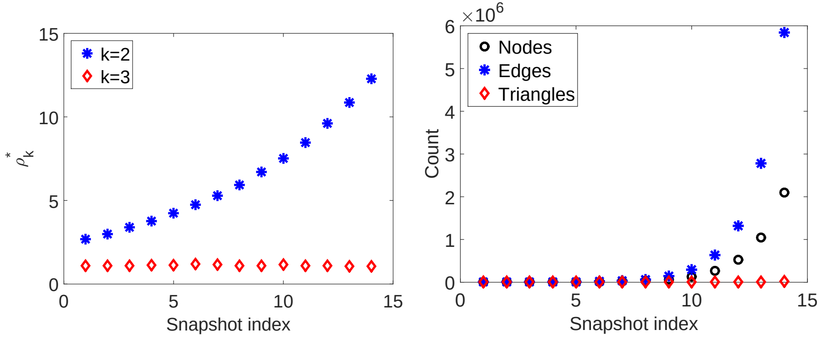
Autonomous systems dataset contains 733 daily instances which span an interval of 785 days from November 8 1997 to January 2 2000



- Despite the average degree increases over time, the optimal density for $k = 2$ remains roughly the same
- The optimal density for $k = 3$ exhibits a mild increasing trend

Time evolving networks

This is how density evolves in stochastic Kronecker graphs with seed matrix [0.9 0.5;0.5 0.2] as we increase the number of nodes as 2^i for $i = 8$ up to $i = 21$



- This and other popular seed matrices can't reproduce what we observe in real-networks with respect to the optimal density

Peeling in batches

The following algorithm due to Bahmani, Vassilvitski and Kumar leads to efficient MapReduce and streaming algorithms [Bahmani et al., 2012]

1. Set $S, \tilde{S} \leftarrow V$
2. **while** $S \neq \emptyset$ **do**
   - $A(S) \leftarrow \{i \in S : D_i(S) \leq 2(1+\epsilon)\rho(S)\}$
   - $S \leftarrow S \backslash A(S)$
   - **if** $\rho(S) \geq \rho(\tilde{S})$ **then** $\tilde{S} \leftarrow S$
3. Return $\tilde{S}$

Peeling in batches

- **Claim**. The previous algorithm achieves a $(2+2\epsilon)$ approximation. Furthermore, it outputs after $\mathcal{O}(\log_{1+\epsilon}(n))$ rounds
- **Proof** .
  - Approximation guarantee: Fix any optimal solution S^*. Consider the first round when a node $v \in S^*$ becomes removed. Let U be the set of vertices at that point. Then, $\rho^* \leq D_v(S^*) \leq D_v(U) \leq (2+2\epsilon)\rho(U)$. QED
  - Number of rounds is $\mathcal{O}(\log_{1+\epsilon}(n))$: The idea is that in each round, we throw away a constant fraction of the vertices $2e(S) > \sum_{v \notin A(S)} D_v(S) > (|S| - |A(S)|)2(1+\epsilon)\rho(S) \rightarrow |A(S)| > \frac{\epsilon}{1+\epsilon}|S| \rightarrow |S| - |A(S)| < \frac{|S|}{1+\epsilon}$

Peeling in batches

Few more remarks

- The previous claim results directly in a $(2+\epsilon)$ approximation algorithm, using $\tilde{O}(n)$ space and $\mathcal{O}(\log n/\epsilon)$
- Similar claim holds for MapReduce. In each round we need to compute degrees and remove $A(S)$
- Many believed that $\mathcal{O}(\log n/\epsilon)$ passes were likely to be necessary
- However, the densest subgraph sparsifier theorem results directly in a **single pass** streaming algorithm that uses $\tilde{O}(n)$ space and provides a $(1+\epsilon)$ approximation guarantee. See also, [Esfandiari et al., 2015, McGregor et al., 2015]

Variations of the DSP

k-densest subgraph	$\delta(S) = \frac{2e[S]}{\|S\|}, \|S\| = k$	**NP**-hard
DalkS	$\delta(S) = \frac{2e[S]}{\|S\|}, \|S\| \geq k$	**NP**-hard
DamkS	$\delta(S) = \frac{2e[S]}{\|S\|}, \|S\| \leq k$	L-reduction to DkS

Densest k subgraph problem

- Does not admit a PTAS unless P=NP
- Feige, Peleg and Kortsarz gave a $\mathcal{O}(n^{\frac{1}{3}})$ approximation algorithm [Feige et al., 2001]
- State of the art algorithm due to Bhaskara et al. provides $\mathcal{O}(n^{\frac{1}{4}+\epsilon})$ approximation guarantee for any $\epsilon > 0$ [Bhaskara et al., 2010]
- Closing the gap between lower and upper bounds is a significant problem

DalkS is **NP**-hard

Proof sketch.

- We reduce the DkS to the DalkS. We are given a graph G and a value k we wish to know whether $\exists S \subseteq V$ such that $\rho(S) \geq \lambda, |S| = k$
- Construct $H = K_{n^2} \cup G$ and run DalkS with lower bound on the number of vertices $n^2 + k$
- Turns out that the part of the optimal DalkS solution on H is the answer to DkS

For the details, see [Khuller and Saha, 2009]

2-approximation for DalkS [Khuller and Saha, 2009]

- The algorithm starts with $G_0 \leftarrow G, D_0 \leftarrow \emptyset$
- In the i-th iteration, we compute the densest subgraph H_i from G_{i-1}
- If $|V(D_{i-1})| + |V(H_i)| \geq k$, terminate
- else
  - $D_i \leftarrow D_{i-1} \cup H_i$
  - Remove H_i from G_{i-1}
  - For every $v \in G_{i-1} \backslash H_i$ add a selfloop of weight w_v where $w_v = |N(v) \cap H_i|$
- When the algorithm stops, each D_i is padded with arbitrary vertices to make their size k, let D_i' be the resulting subgraph
- The algorithm returns the subgraph D_j' with maximum density among the D_i's

2-approximation for DalkS – example

Suppose this is the input to the DalkS

- $k = n + \sqrt{2n}$

- $G = H_1 \cup H_2 \cup H_3 \cup H_4$
  - H_1 is a clique on $\sqrt{2n}$ vertices
  - H_2 is a tree on n vertices
  - H_3 is a cycle on n^2 vertices
  - H_4 is a set of n disjoint vertices

2-approximation for DalkS – example

Let's run the 2-approximation algorithm on G

- First we find H_1 as it is the densest subgraph of G
- In the second iteration it will find H_3
- Therefore, the algorithm has two options:
  - Return $H_1 \cup H_3$
  - Append n arbitrary vertices to H_1. These could well be the n isolated vertices
- In both cases the resulting subgraph has density ≈ 1
- However $H_1 \cup H_2$ has density $\frac{2n}{n+\sqrt{2n}} \approx 2$

Some more remarks

- [Andersen and Chellapilla, 2009] proved that an α approximation for DamkS implies a $\mathcal{O}(\alpha^2)$ approximation algorithm for the DkS
- [Khuller and Saha, 2009] improved this, by showing that an α approximation for DamkS implies a 4α approximation algorithm for the DkS
- The algorithmic ideas we showed for undirected case work for DalkS as well

Efficient algorithms for dynamic graphs

Dynamic setting

We say that an algorithm is a fully-dynamic γ-approximation algorithm for the densest subgraph problem if it can process the following operations.

- $\textsc{Initialize}(n)$: Initialize the algorithm with an empty n-node graph.
- $\textsc{Insert}(u, v)$: Insert edge (u, v) to the graph.
- $\textsc{Delete}(u, v)$: Delete edge (u, v) from the graph.
- $\textsc{QueryValue}$: Output a γ-approximate value of $\rho^*(G) = d^*$

Dynamic setting

The performance of a data structure is measured in term of four different metrics.

- Space-complexity: This is given by the total space (in terms of bits) used by the data structure.
- Update-time: This is the time taken to handle an INSERT or DELETE operation.
- Query-time: This is the time taken to handle a QUERYVALUE operation.
- Preprocessing-time: This is the time taken to handle the INITIALIZE operation. Unless explicitly mentioned otherwise, in this paper the preprocessing time will always be $\tilde{\mathcal{O}}(n)$.

Streaming vs. Dynamic efficiency

- Streaming algorithms' community cares primarily about the space efficiency.
- Dynamic algorithms' community care primarily about the update and query times.
- [Bhattacharya et al., 2015] provide the first result that successfully combines both types of efficiencies simultaneously for the densest subgraph problem
  - Research direction: Can we develop similar type of results for other graph theoretic problems?

$(2+\epsilon)$-approximation 1-pass dynamic semi-streaming algorithm

Theorem ([Bhattacharya et al., 2015])

We can process a dynamic stream of updates in the graph G in $\tilde{\mathcal{O}}(n)$ space, and with high probability return a $(2+\mathcal{O}(\epsilon))$-approximation of $d^* = \max_{S \subseteq V} \rho(S)$ at the end of the stream.

- Remark: To obtain both results we introduce the (α, d, L)-decomposition. It generalizes the well-known d-core, namely the (unique) largest induced subgraph with every node having degree at least d.

(α, d, L)-decomposition – Definition

- Fix any $\alpha \geq 1$, $d \geq 0$, and any positive integer L.
- Consider a family of subsets $Z_1 \supseteq \cdots \supseteq Z_L$.
- The tuple $(Z_1, \ldots, Z_L)$ is an (α, d, L)-decomposition of the input graph $G = (V, E)$ iff:

– $Z_1 = V$ and,

– for every $i \in [L - 1]$, we have

$$Z_{i+1} \supseteq \{v \in Z_i : D_v(Z_i) > \alpha d\}$$

and

$$Z_{i+1} \cap \{v \in Z_i : D_v(Z_i) < d\} = \emptyset.$$

(α, d, L)-decomposition – Key property

Theorem

- *Fix any* $\alpha \geq 1$, $d \geq 0$, $\epsilon \in (0,1)$, $L \leftarrow 2 + \lceil \log_{(1+\epsilon)} n \rceil$.
- *Let* $(Z_1, \ldots, Z_L)$ *be an* (α, d, L)*-decomposition of* $G = (V, E)$.

– *If* $d > 2(1+\epsilon)d^*$, *then* $Z_L = \emptyset$.

– *If* $d < d^*/\alpha$, *then* $Z_L \neq \emptyset$ *and there is an index* $j \in [L]$ *such that* $\rho(Z_j) \geq d/(2(1+\epsilon))$.

Remark 1: A key property of the densest subgraph that prior work [Charikar, 2000] and our work use throughout our work is that $D_v(S^*) \geq d^*$ for any $S^* \subseteq V$ such that $\rho(S^*) = d^*$.
Remark 2: Notice that $\frac{m}{n} \leq d^* < n - 1$.

(α, d, L)-decomposition – Algorithmic aspect

(Rough) Idea of how to turn the previous theorem into an algorithm.

- Discretize the range of d^* as $d_k \leftarrow (1+\epsilon)^{k-1} \cdot \frac{m}{n}$, $k \in [K]$ where $K = \mathcal{O}(\log_{1+\epsilon}(n))$.
- For every $k \in [K]$, construct an (α, d_k, L)-decomposition $(Z_1(k), \ldots, Z_L(k))$, where $L = \mathcal{O}(\log_{1+\epsilon}(n))$.
- Let $k' \leftarrow \max\{k \in [K] : Z_L(k) \neq \emptyset\}$.

Then we have the following guarantees:

1. $d^*/(\alpha(1+\epsilon)) \leq d_{k'} \leq 2(1+\epsilon) \cdot d^*$.
2. There exists an index $j' \in [L]$ such that $\rho(Z_{j'}) \geq d_{k'}/(2(1+\epsilon))$.

$(2+\epsilon)$-approximation 1-pass dynamic semi-streaming algorithm

Our streaming algorithm relies on the fact that if we sample independently each edge with probability (roughly) $\tilde{\mathcal{O}}(\frac{1}{d})$, we can create an (α, d, L)-decomposition whp.

Lemma

Fix a $d > 0$, and let S be a collection of $cm(L-1)\log n/d$ mutually independent simple random samples from the edge-set E of the input graph $G = (V, E)$. With high probability we can construct from S an (α, d, L)-decomposition $(Z_1, \ldots, Z_L)$ of G, using $\tilde{\mathcal{O}}(n)$ bits of space.

$(2+\epsilon)$-approximation 1-pass dynamic semi-streaming algorithm

Emulating Charikar's peeling paradigm.

The algorithm works by partitioning the samples in S evenly among $(L-1)$ groups $\{S_i\}$, $i \in [L-1]$

- Set $Z_1 \leftarrow V$.
- For $i = 1$ to $(L-1)$: Set $Z_{i+1} \leftarrow \{v \in Z_i : D_v(Z_i, S_i) \geq (1-\epsilon)\alpha c \log n\}$.

Here, $D_v(Z_i, S_i)$ is the number of neighbors of v in set Z_i connected through the set of edges S_i.

$(2+\epsilon)$-approximation 1-pass dynamic semi-streaming algorithm

- "Guess" the number of edges m.
- For each guess of m, build $\mathcal{O}(\log n/\epsilon)$ $(\alpha, d_k = (1+\epsilon)^{k-1}\frac{m}{n}, L)$-decompositions, one for each density guess d_k. Set $\alpha = \frac{1+\epsilon}{1-\epsilon}$.
- For each guess of d_k maintain a sample S of $cm(L-1)\log n/d_k = \tilde{\mathcal{O}}(n)$ random edges.
- Perform peeling and find k'.

Few remarks.

1. The case of dynamic streams is dealt with by using ℓ_0 samplers [Jowhari et al., 2011].
2. For the dynamic case, we wish to find an α large enough to be lazy enough when we update our data structures, small enough to achieve a good approximation.

Fully dynamic $(4+\epsilon)$-approximation algorithm $\tilde{\mathcal{O}}(n)$ space

Theorem ([Bhattacharya et al., 2015])

- *Let* $\epsilon \in (0,1)$, $\lambda > 1$ *constant and* $T = \lceil n^{\lambda} \rceil$.
- *There is an algorithm that processes the first* T *updates in the dynamic stream such that:*

– *It uses* $\tilde{\mathcal{O}}(n)$ *space (Space efficiency)*

– *It maintains a value* $\textsc{Output}^{(t)}$ *at each* $t \in [T]$ *such that for all* $t \in [T]$ *whp*

$$\textsc{Opt}^{(t)}/(4+\Theta(\epsilon)) \leq \textsc{Output}^{(t)} \leq \textsc{Opt}^{(t)}.$$

Also, the total amount of computation performed while processing the first T *updates in the dynamic stream is* $\mathcal{O}(T \,\mathrm{poly} \log n)$. *(Time efficiency)*

Fully dynamic $(4+\epsilon)$-approximation algorithm $\mathcal{O}(n+m)$ space

- As before, we discretize the range of d^* in the same way, i.e., in powers of $(1+\epsilon)$ by defining the values $\{d_k\}, k \in [K]$.
- For each d_k we are able to maintain an (α, d_k, L)-decomposition of G in time $\mathcal{O}(L/\epsilon) = \mathcal{O}(\log n/\epsilon^2)$ per edge update.
- The total time for all K decompositions is $\mathcal{O}(\log^2 n/\epsilon^3)$ per update operation.
- Remark: We find an α large enough to be lazy enough, small enough to achieve a good approximation. It turns out using a fine tuned potential function analysis, that for $\alpha = 2 + \Theta(\epsilon)$ we achieve good amortized time and a $(4 + \Theta(\epsilon))$-approximation.

Remark: How to maintain efficiently a random sample of $\tilde{\mathcal{O}}(n)$ edges when the graph changes?

Q1 How do we maintain dynamically the random sample(s) of $\tilde{\mathcal{O}}(n)$ edges?

– If we naively run an ℓ_0 sampler responsible for an edge in the sample for each update, we need $\tilde{\mathcal{O}}(n)$ time per update.

Idea: When an update takes place, only one ℓ_0 sampler needs to be invoked. Let $E = \binom{[n]}{2} \supseteq E^{(t)}$.

- Let $h : E \to [s_k]$ be an ℓ-wise independent hash function
- The i-th "bucket" $Q_i^{(t)}$ is responsible for all edges such that $h(e) = i$, for each $i = 1, \ldots, s_k$. We also run an independent copy of an ℓ_0 sampler.

Few more remarks

- To make Chernoff+union bound work we need $l = \tilde{\mathcal{O}}(n)$. To construct our hash function we invoke the construction due to [Pagh and Pagh, 2008].
- The previous theorem [Bhattacharya et al., 2015] opens the direction towards single-pass semi-streaming algorithms over dynamic streams with polylogarithmic update and query times.
- [Epasto et al., 2015] provided a $(2+\epsilon)$-approximation algorithm, $\mathcal{O}(polylog(n)) = \tilde{\mathcal{O}}(1)$ amortized time per update, $\mathcal{O}(n+m)$ space under the assumption that deletions are *random*.

Problem variants

Problem variants II : top-k dense subgraphs

Top-k dense subgraphs

- in many cases we want to find more than one dense subgraph
- idea: find all dense subgraphs (e.g., denser than a threshold)
- cut enumeration techniques to output all near-optimal dense subgraphs ([Saha et al., 2010])
- in practice, this method suffers from output degeneracies:
  - many subsets of a dense subgraph tend to be near-optimally dense as well

Top-k dense subgraphs

- another approach
  (i) find a dense subgraph S
  (ii) remove all vertices and edges of S
  (iii) iterate
- reported subgraphs are disjoint
- certain degree of overlap can be desirable [Balalau et al., 2015]

Top-k dense subgraphs with limited overlap

problem formulation ([Balalau et al., 2015])

- given graph $G = (V, E)$, and parameters k and α
- find k subgraphs $S_1, \ldots, S_k$
- in order to maximize

$$\sum_{i=1}^{k} d(S_i)$$

subject to

$$\frac{|S_i \cap S_j|}{|S_i \cup S_j|} \leq \alpha, \text{ for all } 1 \leq i < j \leq k$$

Top-k dense subgraphs with limited overlap

algorithm MinAndRemove ([Balalau et al., 2015])

input: undirected graph $G = (V, E)$, parameters k and α
output: k subgraphs $G_1, \ldots, G_k$ with overlap at most α

while less than k subgraphs found and G non-empty
    find minimal densest subgraph $G_i = (V_i, E_i)$
    for each $v \in V_i$
        $\Delta_G(v) \leftarrow$ the set of neighbors of v in G
    remove $\lceil (1-\alpha)|V_i| \rceil$ nodes with minimum $|\Delta_G(v) \setminus V_i|$
        and all their edges from G

Top-k dense subgraphs with limited overlap

summary of results ([Balalau et al., 2015])

- MinAndRemove finds optimal solution,
  if this contains disjoint subgraphs
- MinAndRemove works shown to work well in practice
- faster algorithm, at small loss of accuracy

Problem variants III : core decomposition

k-core decomposition

widely used technique for partitioning graphs

$$k\text{-core} = \text{largest subgraph with vertex degrees} \geq k$$

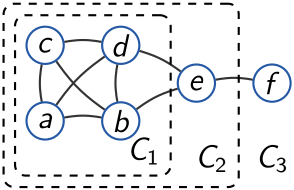
cores form a chain, k-core $\subseteq (k-1)$-core; let

$$k\text{-shell} = \text{vertices in } k\text{-core but not in } (k+1)\text{-core}$$

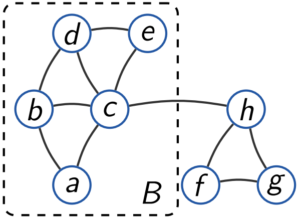
algorithm to find shells:

1. **while** G is not empty
2. $\quad v \leftarrow$ vertex with the smallest degree
3. $\quad$ assign v to k-shell
4. $\quad$ remove v from G

core decomposition and density are not compatible



$d(C_1) = \frac{6}{4} < \frac{8}{5} = d(C_2)$



only one core but
$d(B) = \frac{7}{5} > \frac{11}{8} = d(G)$

density-friendly decomposition

goal:

adapt k-core decomposition for density

obtain a nested sequence of increasingly dense subgraphs

[Tatti and Gionis, 2015]

locally-dense subgraphs

informally,

subgraph H is locally-dense = any subgraph of H is denser than any subgraph outside H

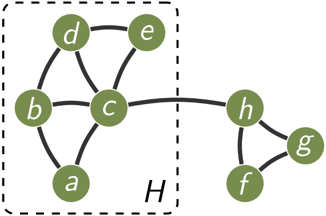
formally, define augmented density

$$d(X, Y) = \frac{|E(X)| + |E(X, Y)|}{|X|}, \quad \text{for} \quad X \cap Y = \emptyset$$

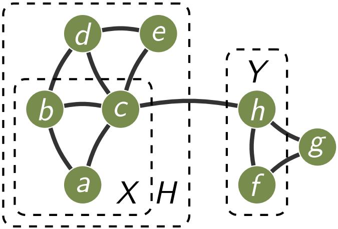
subgraph H is locally-dense if

$$d(X, H \setminus X) > d(Y, H), \quad \text{for any} \quad X \subsetneq H, Y \cap H = \emptyset$$

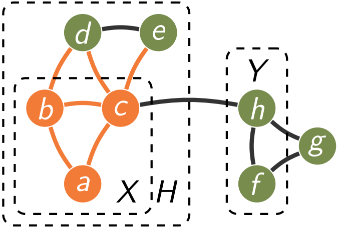
example



example

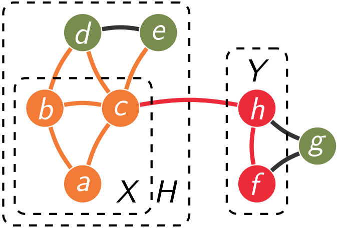


example



$d(X, H \setminus X) = 6/3$

example



$$d(X, H \setminus X) = 6/3$$
$$d(Y, H) = 2/2$$

properties

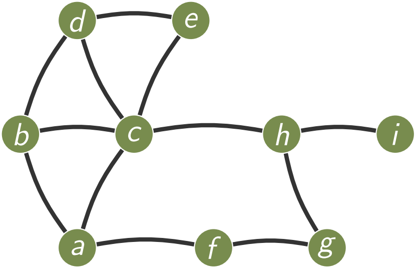
locally-dense subgraphs form a chain

$$\emptyset = B_0 \subsetneq B_1 \subsetneq B_2 \subsetneq \cdots \subsetneq B_k = G$$

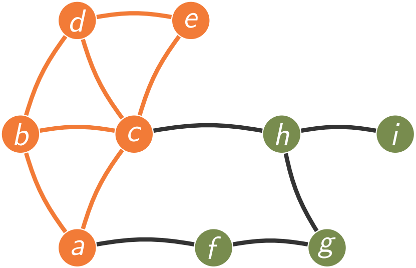
B_i is the densest subgraph containing B_{i-1}

$$\begin{aligned} B_1 &= \text{densest subgraph} \\ B_2 &= \arg\max_{B \supsetneq B_1} d(B \setminus B_1, B_1) \\ &\ldots \\ B_i &= \arg\max_{B \supsetneq B_{i-1}} d(B \setminus B_{i-1}, B_{i-1}) \end{aligned}$$

first approach to compute the subgraphs

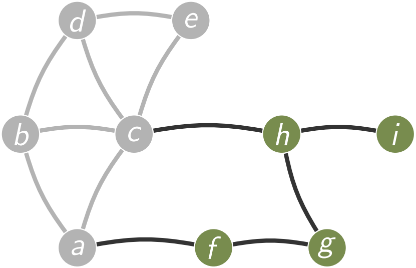


first approach to compute the subgraphs



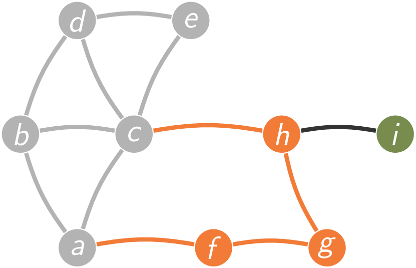
find B_1

first approach to compute the subgraphs



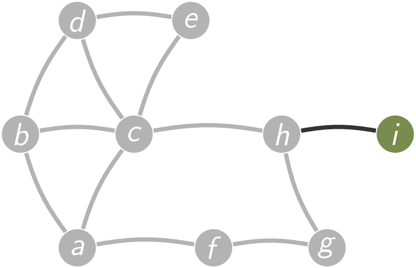
find B_1
delete B_1

first approach to compute the subgraphs



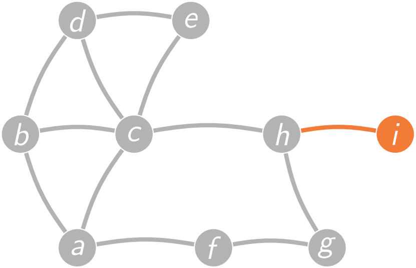
find B_1
delete B_1
find B_2

first approach to compute the subgraphs



find B_1
delete B_1
find B_2
delete B_2

first approach to compute the subgraphs



find B_1
delete B_1
find B_2
delete B_2
find B_3

computing the subgraphs

define

$$F(\alpha) = \arg\max_X |E(X)| - \alpha|X|$$

Goldberg showed that

- $F(\alpha)$ can be solved with a min-cut
- there is α such that $F(\alpha)$ is the densest subgraph

we can show that

- $F(\alpha)$ is locally-dense
- for every B_i there is α such that $B_i = F(\alpha)$

computing the subgraphs

find all B_i by varying α (with divide-and-conquer)

algorithm: $\textsc{Exact}(X, Y)$

1. select α such that $X \subseteq F(\alpha) \subsetneq Y$
2. $Z \leftarrow F(\alpha)$
2. **if** $(Z \neq X)$
3. **output** Z
3. $\textsc{Exact}(X, Z)$
3. $\textsc{Exact}(Z, Y)$

- we need only $2k - 3$ calls of $F(\alpha)$
  (k is the number of locally-dense subgraphs)
- $O(n^2 m)$ total running time, in practice much faster
- $X \subset F(\alpha) \subset Y$ allows optimizations

approximation with profiles

approximation guarantees are tricky:

- algorithm may return different number of subgraphs

define a profile:

$$p(i; \mathcal{B}) = \begin{cases} d(B_1) & \text{if } i \leq |B_1| \\ d(B_2 \setminus B_1, B_1) & \text{if } |B_1| < i \leq |B_2| \\ \ldots \end{cases}$$

core decomposition

let $\mathcal{C}$ be the core decomposition

let $\mathcal{B}$ be the optimal locally-dense decomposition

then

$$p(i; \mathcal{C}) \geq p(i; \mathcal{B})/2, \quad \text{for every} \quad i$$

for $i = 1$, this implies

$$d(C_1) \geq d(B_1)/2$$

extending Charikar's algorithm

$C_1 \leftarrow$ densest subgraph of form $v_1, \ldots v_{|C_1|}$

$C_2 \leftarrow$ subgraph maximizing $d(v_1, \ldots v_{|C_2|} \setminus C_1, C_1)$

$C_3 \leftarrow$ subgraph maximizing $d(v_1, \ldots v_{|C_3|} \setminus C_2, C_2)$

$\ldots$

The graphs C_i

- can be found in $O(n^2)$-time naively
- can be found in $O(n)$-time with PAV algorithm [Ayer et al., 1955]

greedy decomposition

let $\mathcal{C}$ be the greedy decomposition

(found by the extension of Charikar's algorithm)

let $\mathcal{B}$ be the optimal locally-dense decomposition

then

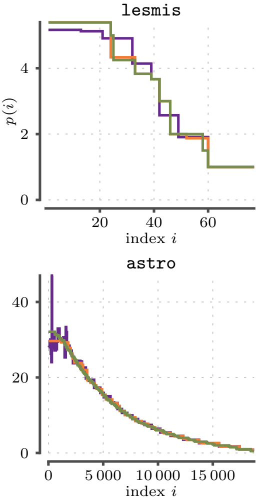
$$p(i; \mathcal{C}) \geq p(i; \mathcal{B})/2, \quad \text{for every} \quad i$$

for $i = 1$, this implies

$$d(C_1) \geq d(B_1)/2$$

experiments

how well these algorithm perform?



summary (density-friendly decomposition)

- decomposition based on average density
- can be computed exactly in $\mathcal{O}(n^2 m)$ time, faster in practice
- can be $1/2$-approximated in linear time by
  - k-core decomposition
  - greedy algorithm

future work:

- consider different density functions
- control the size of the decomposition

Problem variants IV : community search

community detection problems

- typical problem formulations require non-overlapping and complete partition of the set of vertices
- quite restrictive
- inherently ambiguous: research group vs. bicycling club

- additional information can resolve ambiquity
- community defined by two or more people

the community-search problem

- given graph $G = (V, E)$, and
- given a subset of vertices $Q \subseteq V$ (the query vertices)
- find a community H that contains Q

applications

- find the community of a given set of users (cocktail party)
- recommend tags for an image (tag recommendation)
- form a team to solve a problem (team formation)

center-piece subgraph

[Tong and Faloutsos, 2006]

- given: graph $G = (V, E)$ and set of query vertices $Q \subseteq V$
- find: a connected subgraph H that
  (a) contains Q
  (b) optimizes a goodness function $g(H)$

- main concepts:
- `k_softAND`: a node in H should be well connected to at least k vertices of Q
- $r(i, j)$ goodness score of j wrt $q_i \in Q$
- $r(Q, j)$ goodness score of j wrt Q
- $g(H)$ goodness score of a candidate subgraph H
- $H^* = \arg\max_H g(H)$

center-piece subgraph

[Tong and Faloutsos, 2006]

- $r(i, j)$ goodness score of j wrt $q_i \in Q$

  probability to meet j in a random walk with restart to q_i

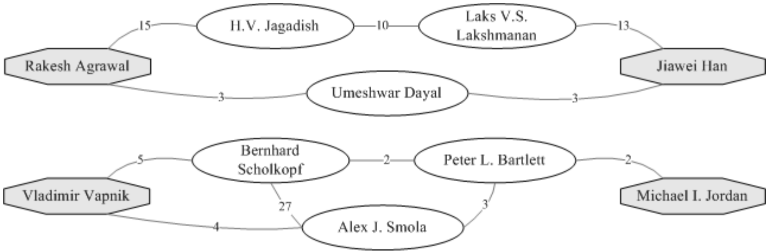
- $r(Q, j)$ goodness score of j wrt Q

  probability to meet j in a random walk with restart to k vertices of Q

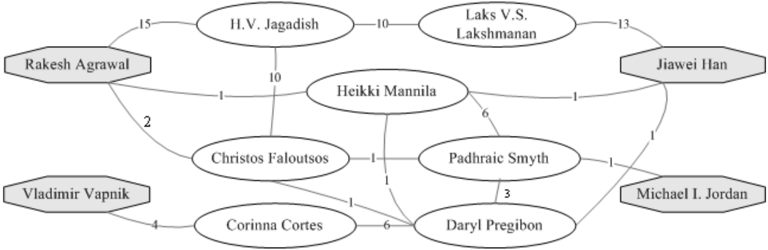
- proposed algorithm:

1. greedy: find a good destination vertex j ito add in H
2. add a path from each of top-k vertices of Q path to j
3. stop when H becomes large enough

center-piece subgraph — example results



(a) “K_softANDquery”: $k = 2$



the community-search problem

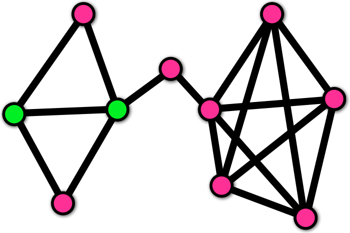
- given: graph $G = (V, E)$ and set of query vertices $Q \subseteq V$
- find: a connected subgraph H that
  (a) contains Q
  (b) optimizes a density function $d(H)$
  (c) possibly other constraints

- density function (b):

  average degree, minimum degree, quasiclique, etc.

  measured on the induced subgraph H

free riders



- remedy 1: use min degree as density function
- remedy 2: use distance constraint

$$d(Q,j) = \sum_{q \in Q} d^2(q_i, j) \leq B$$

the community-search problem

adaptation of the greedy algorithm of [Charikar, 2000]

input: undirected graph $G = (V, E)$, query vertices $Q \subseteq V$
output: connected, dense subgraph H

1 set $G_n \leftarrow G$
2 for $k \leftarrow n$ downto 1
2.1 remove all vertices violating distance constraints
2.2 let v be the smallest degree vertex in G_k among all vertices not in Q
2.3 $G_{k-1} \leftarrow G_k \setminus \{v\}$
2.4 if left only with vertices in Q or disconnected graph, stop
3 output the subgraph in $G_n, \ldots, G_1$ that maximizes $f(H)$

properties of the greedy algorithm

- returns optimal solution if no size constraints
- upper-bound constraints make the problem **NP**-hard (heuristic solution, also adaptation of the greedy)
- generalization for monotone constraints and monotone objective functions

experimental evaluation (qualitative summary)

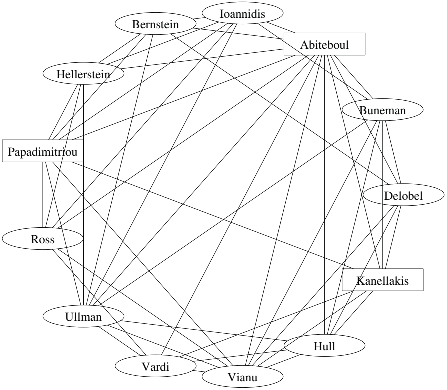
baseline: increamental addition of vertices

- start with a Steiner tree on the query vertices
- greedily add vertices
- return best solution among all solutions constructed

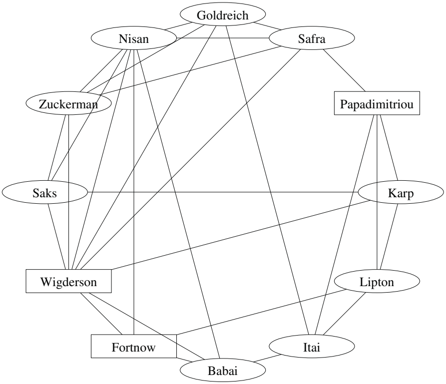
example result in DBLP

- proposed algorithm: min degree $= 3$, avg degree $= 6$
- baseline algorithm: min degree $= 1.5$, avg degree $= 2.5$

the community-search problem — example results



(a) Database theory



(b) Complexity theory

(from [Sozio and Gionis, 2010])

monotone functions

function f is monotone non-increasing if

for every graph G and

for every subgraph H of G it is

$$f(H) \leq f(G)$$

the following functions are monotone non-increasing:

- the query nodes are connected in H (0/1)
- are the nodes in H able to perform a set of tasks?
- upper-bound distance constraint
- lower-bound constraint on the size of H

generalization to monotone functions

generalized community-search problem

given

- a graph $G = (V, E)$
- a node-monotone non-increasing function f
- $f_1, \ldots, f_k$ non-increasing boolean functions

find

- a subgraph H of G
- satisfying $f_1, \ldots, f_k$ and
- maximizing f

generalized greedy

1 set $G_n \leftarrow G$
2 for $k \leftarrow n$ downto 1
2.1 remove all vertices violating any constraint $f_1, \ldots, f_k$
2.2 let v minimizing $f(G_k, v)$
2.3 $G_{k-1} \leftarrow G_k \setminus \{v\}$
3 output the subgraph H in $G_n, \ldots, G_1$ that maximizes $f(H, v)$

generalized greedy

theorem

generalized greedy computes an optimum solution
for the generalized community-search problem

running time

- depends on the time to evaluate the functions $f_1, \ldots, f_k$
- formally $\mathcal{O}(m + \sum_i nT_i)$
- where T_i is the time to evaluate f_i

Problem variants V : heavy subgraphs

discovering heavy subgraphs

- given a graph $G = (V, E, d, w)$
  with a distance function $d : E \rightarrow \mathbb{R}$ on edges
  and weights on vertices $w : V \rightarrow \mathbb{R}$
- find a subset of vertices $S \subseteq V$
  so that

1. total weight in S is high
2. vertices in S are close to each other

[Rozenshtein et al., 2014a]

discovering heavy subgraphs

- what does total weight and close to each other mean?
- total weight

$$W(S) = \sum_{v \in S} w(v)$$

- close to each other

$$D(S) = \sum_{u \in S} \sum_{v \in S} d(u, v)$$

- want to maximize $W(S)$ and minimize $D(S)$
- maximize

$$Q(S) = \lambda W(S) - D(S)$$

applications of discovering heavy subgraphs

- finding events in networks
- vertices correspond to locations
- weights model activity recorded in locations
- distances between locations
- find compact regions (neighborhoods) with high activity

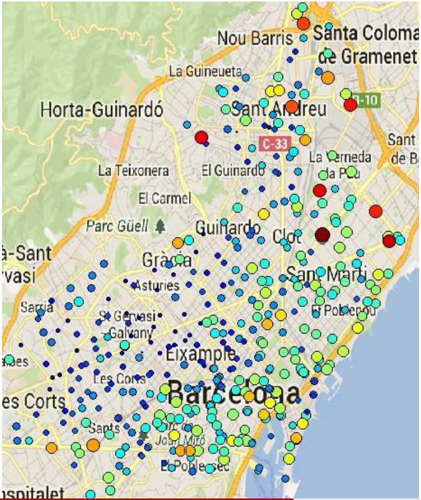
event detection

- sensor networks and traffic measurements

event detection

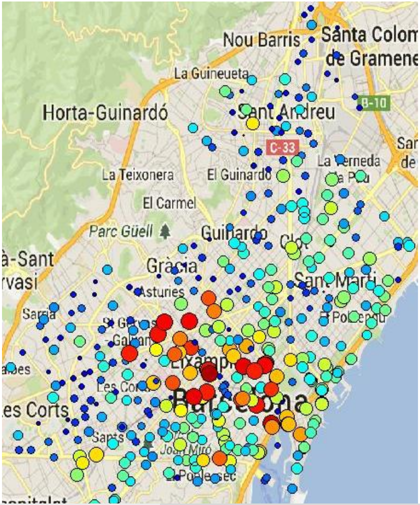
15.11.2012

ordinary day, no events



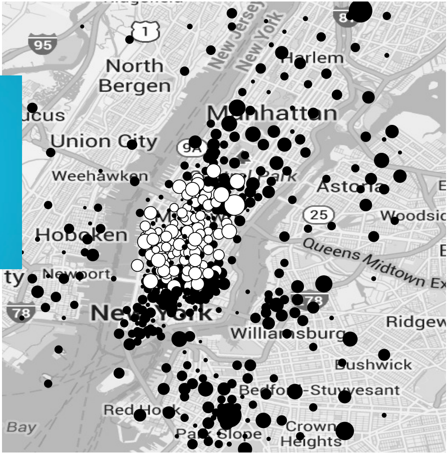
11.09.2012

Catalunya national day



event detection

- location-based social networks



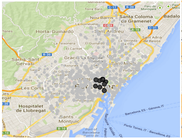
discovering heavy subgraphs

- maximize $Q(S) = \lambda W(S) - D(S)$
- objective can by negative
- add a constant term to ensure non-negativity
- maximize $Q(S) = \lambda W(S) - D(S) + D(V)$

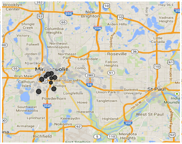
discovering heavy subgraphs

- maximize $Q(S) = \lambda W(S) - D(S) + D(V)$
- objective is submodular (but not monotone)
- can obtain $\frac{1}{2}$-approximation guarantee [Buchbinder et al., 2012]
- problem can be mapped to the max-cut problem which gives 0.868-approximation guarantee [Rozenshtein et al., 2014a]

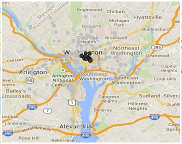
events discovered with bicing and 4square data



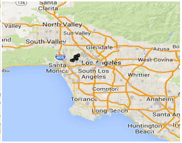
(a) Barcelona: 11.09.12 National Day of Catalonia



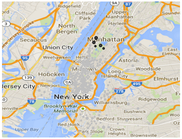
(b) Minneapolis: 4.07.12 Independence Day



(c) Washington, DC: 27.05.13 Memorial Day

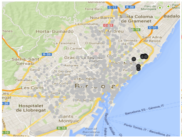


(d) Los Angeles: 31.05.10 Memorial Day

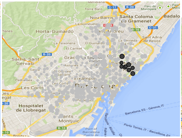


(e) New York: 6.09.10 Labor Day

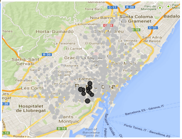
Figure 4: Public holiday city-events discovered using the SDP algorithm.



(a) 01.06.12 Primavera sound music festival



(b) 18.09.12 festival of the Poblenou neighborhood



(c) 31.10.12 Halloween

Problem variants VI :

dense subgraphs in interaction networks

dense subgraphs in interaction networks

- interaction networks : networks with temporal information
  - phonecall networks
  - SMS networks
  - email networks
  - conversation in social-media platforms

- hypothesis : analysis of temporal information
  can reveal hidden structure

[Rozenshtein et al., 2014b]

problem formulation

- given interaction network $G = (V, E)$
- where edges $E = \{(u, v, t)\}$ have time-stamps
- find

  subset of vertices $S \subseteq V$, and

  set T of k time intervals of bounded length

- so that the subgraph induced by S and projected in T is as dense as possible

iterative approach

- decompose the problem in two subproblems
  1. given fixed set of intervals find densest subgraph
  2. given fixed set of vertices find optimal set of intervals
- iterate until convergence

the two subproblems

- subproblem 1 : find optimal vertices given intervals
  - standard densest subgraph problem
  - use the algorithms of Goldberg, or Charikar, etc.

- subproblem 2 : find optimal intervals given vertices
  - NP-hard problem
  - develop greedy heuristic based on the generalized maximum coverage problem
    - iteratively add k intervals
    - select a new interval to maximize density per unit of time
  - due to concavity property searching the next interval can be done in linear time

sample experimental results — enron email network

dataset

Name	$\lvert V \rvert$	$\lvert \pi(E) \rvert$	$\lvert E \rvert$	$\lvert T \rvert$	$d(\pi(G))$	$d(H)$
Enron	1143	2019	6245	8080	3.53	14.38

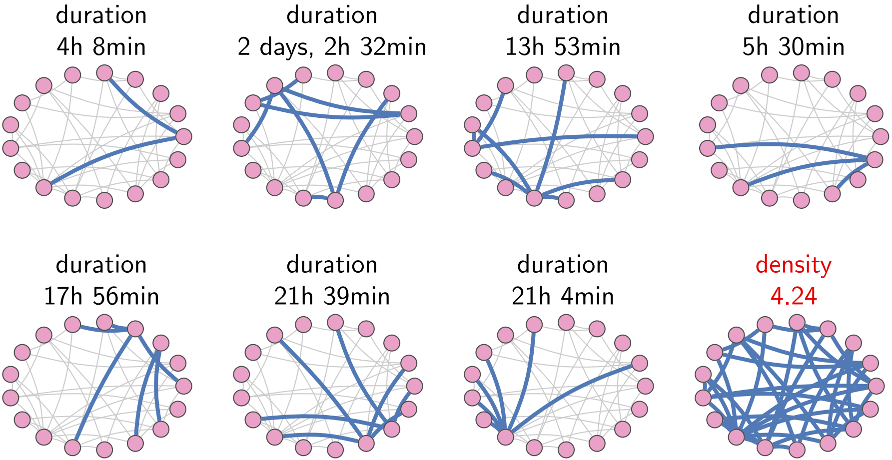
dynamic dense subgraphs

			Community density			Community size		
Dataset	B	K	GA	BA	BASE	GA	BA	BASE
Enron	1	1	6.18	6.18	6.18	11	11	11
		5	10	10.37	6.18	17	16	11
		10	12.2	12.38	6.18	20	21	11
	7	1	6.36	6.36	6.36	11	11	11
		5	11.26	11.23	6.36	19	26	11
		10	13.07	13.07	6.36	28	28	11

sample experimental results — twitter network

Method	Size	Density	Hashtags
GA	9	4.9	aaltoes, startup, vc, summerofstartups, web, startups, entrepreneur, slush10, skype, funrank, africa, mobile, demoday, design, linkedin, aalto

sample experimental results — facebook network



Open problems

Open problems I

- can we improve the $(4 + \epsilon)$ approximation guarantee?
- what about weighted graphs?
- polylogarithmic worst-case update time?
- space- and time-efficient fully dynamic algorithm for other graph problems, e.g., single-source shortest paths?
  - remark: for the connectivity problem, one can combine the space-efficient streaming algorithm of [Ahn et al., 2012] with the fully-dynamic algorithm of [Kapron et al., 2013]

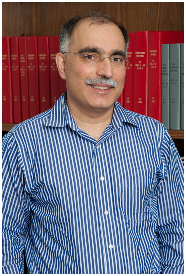
Open problems II

- improve lower bounds for dynamic case [Henzinger et al., 2015]
- for which graph problems does uniform sampling result in high-quality approximation?
  - triangle sparsifiers [Tsourakakis et al., 2011]
  - densest subgraphs [Bhattacharya et al., 2015], [Mitzenmacher et al., 2015]
  - d-max cut, d-sum max clustering [Esfandiari et al., 2015]
  - main difficulty: Chernoff + union bound does not work because of exponential number of bad events

Open problems III

- further study of top-k densest subgraph problem, and develop approximation guarantees
- incorporate temporal and/or spatial information
  application: finding local events in social networks
- dense subgraphs with query nodes in graph streams
  preprocessing vs. query-time processing trade-off
- incorporate developed techniques into real-time analytics systems
- deploy existing tools on more real-world applications
  (for code see `https://github.com/tsourolampis`)

Acknowledgements



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