Aalto University
School of Science

Algorithmic methods for mining large graphs
Lecure 3 : Finding dense subgraphs

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course agenda

• introduction to graph mining	Tue afternoon
• computing basic graph statistics	Tue afternoon, Wed morning
• finding dense subgraphs	Wed afternoon, Thu morning
• spectral graph analysis	Thu afternoon
• additional topics	Fri morning
– inferring hierarchies in graphs	
– mining dynamic graphs	
– graph sparsifiers	

what this lecture is about . . .

given a graph (network), static or dynamic

(social network, biological network, information network, . . .)

find a subgraph that . . .

. . . has many edges

. . . is densely connected

why I care?

what does dense mean?

review of main problems, and main algorithms

outline

- motivating applications
- preliminaries and measures of density
- algorithms for finding dense subgraphs
- problem variants

motivating applications

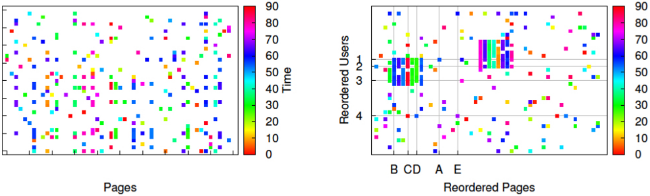
motivation – correlation mining

correlation mining: a general framework with many applications

- data is converted into a graph
- vertices correspond to entities
- an edge between two entities denotes strong correlation
  1. stock correlation network: data represent stock timeseries
  2. gene correlation networks: data represent gene expression
- dense subsets of vertices correspond to highly correlated entities
- applications:
  1. analysis of stock market dynamics
  2. detecting co-expression modules

motivation – fraud detection

- dense bipartite subgraphs in page-like data reveal attempts to inflate page-like counts [Beutel et al., 2013]



source: [Beutel et al., 2013]

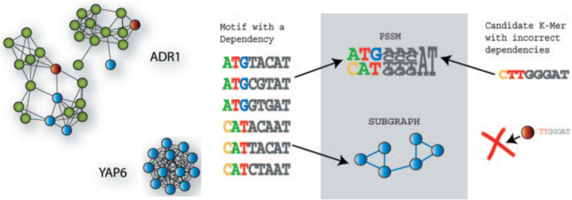
motivation – e-commerce

e-commerce

- weighted bipartite graph $G(A \cup Q, E, w)$
- set A corresponds to advertisers
- set Q corresponds to queries
- each edge (a, q) has weight $w(a, q)$ equal to the amount of money advertiser a is willing to spend on query q

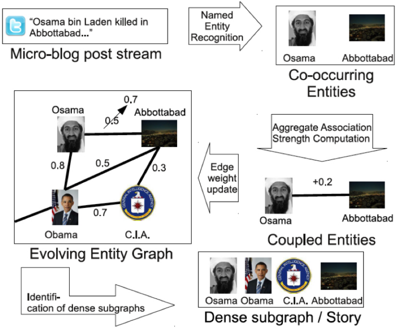
large almost bipartite cliques correspond to sub-markets

motivation – bioinformatics



- DNA motif detection [Fratkin et al., 2006]
  - vertices correspond to *k*-mers
  - edges represent nucleotide similarities between *k*-mers
- gene correlation analysis
- detect complex annotation patterns from gene annotation data [Saha et al., 2010]

motivation – mining twitter data



real-time story identification [Angel et al., 2012]

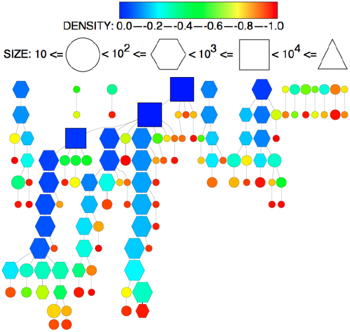
- mining of twitter data
- vertices correspond to entities
- edges correspond to co-occurence of entities
- dense subgraphs capture news stories

motivation – graph mining

understanding the structure of real-world networks

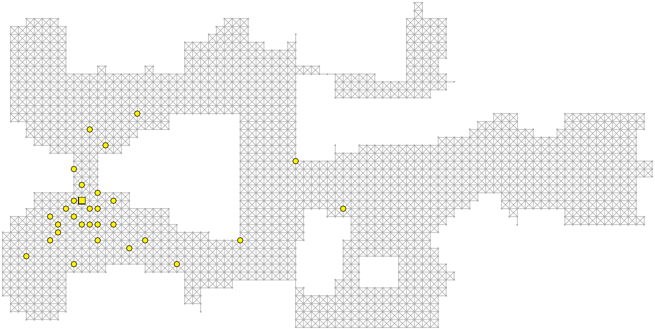
[Sarıyüce et al., 2015]

nucleus decomposition of a graph



(3,4)-nuclei forest for facebook

motivation – distance queries in graphs

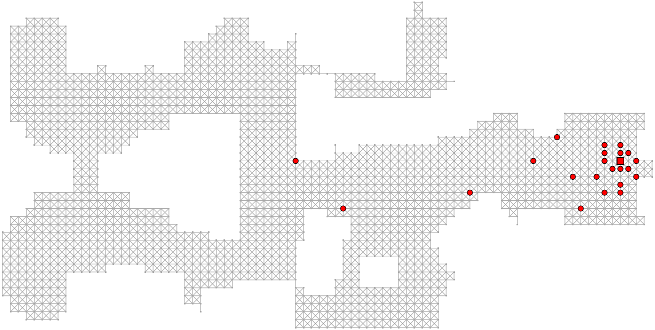


- $L(u) \equiv$ set of pairs $(v, \text{dist}(u, v))$

  $L(u)$ is the *label* of u; each v is a *hub* for u.

figure from [Delling et al., 2014]

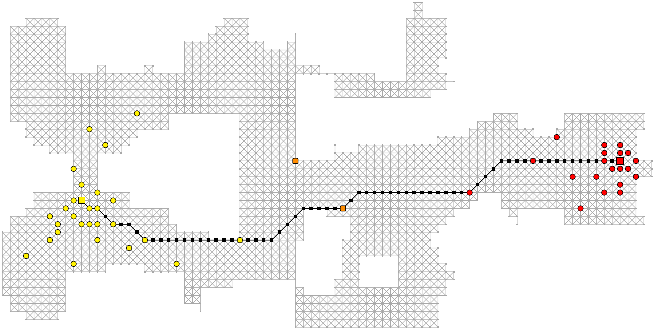
motivation – distance queries in graphs



- preprocessing : compute a label set for every vertex
- cover property : for all s, t intersection $L(s) \cap L(t)$ must hit an s–t shortest path

figure from [Delling et al., 2014]

motivation – distance queries in graphs



- to answer an s–t query :

  find hub v in $L(s) \cap L(t)$ minimizing $\mathrm{dist}(s, v) + \mathrm{dist}(v, t)$

figure from [Delling et al., 2014]

motivation – distance queries in graphs

hub label queries are trivial to implement :

- entries sorted by hub id
- linear sweep to find matches
- access to only two contiguous blocks (cache-friendly)

method is practical if labels sets are small

- can we find small labels sets?
- 2-hop labeling algorithm relies on dense-subgraph discovery to find such label sets (!) [Cohen et al., 2003]
- state-of-art 2-hop labeling scheme : [Delling et al., 2014]
- more work on the topic : [Peleg, 2000, Thorup, 2004]

motivation – frequent pattern mining

- given a set of transactions over items
- find item sets that occur together in a θ fraction of the transactions

issue number	heroes
1	Iceman, Storm, Wolverine
2	Aurora, Cyclops, Magneto, Storm
3	Beast, Cyclops, Iceman, Magneto
4	Cyclops, Iceman, Storm, Wolverine
5	Beast, Iceman, Magneto, Storm

e.g., {Iceman, Storm} appear in 60% of issues

motivation – frequent pattern mining

- one of the most well-studied area in data mining
- many efficient algorithms

  Apriori, Eclat, FP-growth, Mafia, ABS, ...
- main idea: monotonicity

  a subset of a frequent set must be frequent, or

  a superset of an infrequent set must be infrequent
- algorithmically:

  start with small itemsets

  proceed with larger itemset if all subsets are frequent
- enumerate all frequent itemsets

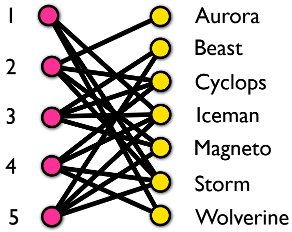
motivation – frequent itemsets and dense subgraphs

id	heroes
1	Iceman, Storm, Wolverine
2	Aurora, Cyclops, Magneto, Storm
3	Beast, Cyclops, Iceman, Magneto
4	Cyclops, Iceman, Storm, Wolverine
5	Beast, Iceman, Magneto, Storm

⇔

	A B C I M S W
1	0 0 0 1 0 1 1
2	1 0 1 1 1 0 0
3	0 1 1 1 1 0 0
4	0 0 1 1 0 1 1
5	0 1 0 1 1 1 0

⇔



- transaction data ⇔ binary data ⇔ bipartite graphs

motivation – frequent itemsets and dense subgraphs

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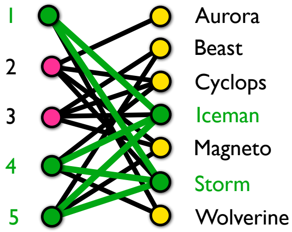


- transaction data ⇔ binary data ⇔ bipartite graphs
- frequent itemsets ⇔ bi-cliques

motivation – finding web communities

[Kumar et al., 1999]

- hypothesis: web communities consist of hub-like pages and authority-like pages

  e.g., luxury cars and luxury-car aficionados

- key observations:

1. let $G = (U, V, E)$ be a dense web community

   then G should contain some small core (bi-clique)

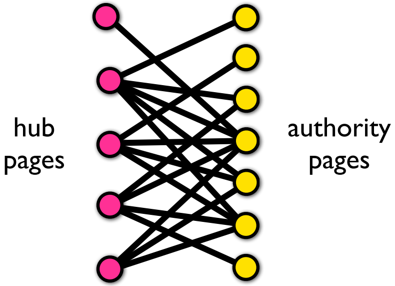
2. consider a web graph with no communities

   then small cores are unlikely

- both observations motivated from theory of random graphs

motivation – finding web communities

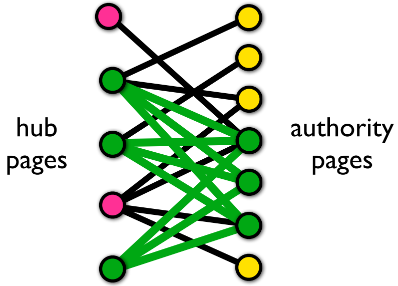
a web community



[Kumar et al., 1999]

motivation – finding web communities

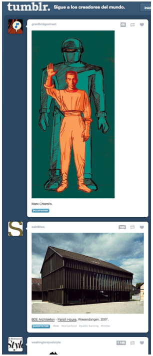
web communities containts small cores

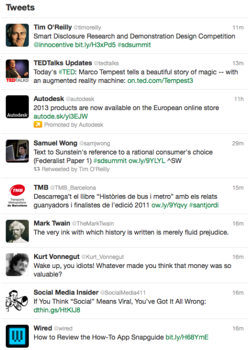


[Kumar et al., 1999]

motivation – social piggybacking

[Gionis et al., 2013]

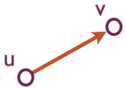




- event feeds: majority of activity in social networks

motivation – social piggybacking

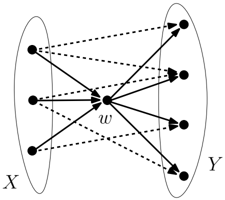
- system throughput proportional to the data transferred between data stores
- feed generation important component to optimize



- primitive operation: transfer data between two data stores
- can be implemented as push or pull strategy
- optimal strategy depends on production and consumption rates of nodes

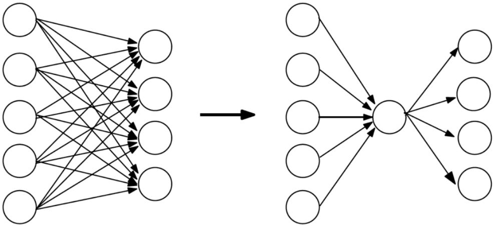
motivation – social piggybacking

- hub optimization turns out to be a good idea
- depends on finding dense subgraphs



motivation – graph compression

- compress web graphs by finding and compressing bi-cliques [Karande et al., 2009]
- many graph mining tasks that can be formulated as matrix-vector multiplication are more efficient on the compressed graph [Kang et al., 2009]



motivation – more applications

- graph visualization [Alvarez-Hamelin et al., 2005]
- community detection [Chen and Saad, 2012]
- epilepsy prediction [Iasemidis et al., 2003]
- event detection in activity networks [Rozenshtein et al., 2014]
- many more

landscape of related work

- brute force [Johnson and Trick, 1996]
- heuristics [Bomze et al., 1999]
  - spectral algorithms [Alon et al., 1998, McSherry, 2001, Papailiopoulos et al., 2014]
  - belief-propagation methods [Kang et al., 2011]
- enumerating maximal cliques, e.g., [Bron and Kerbosch, 1973, Eppstein et al., 2010, Makino and Uno, 2004]
- NP-hard formulations and various relaxations
  - maximum clique problem [Karp, 1972, Hastad, 1999]
  - *k*-densest subgraph problem [Bhaskara et al., 2010, Feige et al., 2001]
  - optimal quasi-cliques [Tsourakakis et al., 2013]
- polynomial-time solvable objectives
  - densest subgraph problem [Goldberg, 1984]
  - *"The densest subgraph problem lies at the core of large scale data mining"* [Bahmani et al., 2012]

preliminaries, measures of density

notation

- graph $G = (V, E)$ with vertices V and edges $E \subseteq V \times V$
- degree of a node $u \in V$ with respect to $X \subseteq V$ is

$$\deg_X(u) = |\{v \in X \text{ such that } (u, v) \in E\}|$$

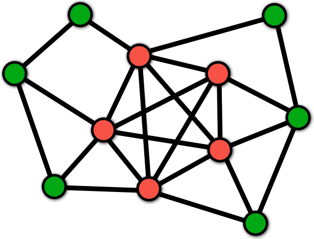
- degree of a node $u \in V$ is $\deg(u) = \deg_V(u)$
- edges between $S \subseteq V$ and $T \subseteq V$ are

$$E(S, T) = \{(u, v) \text{ such that } u \in S \text{ and } v \in T\}$$

  use shorthand $E(S)$ for $E(S, S)$
- graph cut is defined by a subset of vertices $S \subseteq V$
- edges of a graph cut $S \subseteq V$ are $E(S, \bar{S}) = E(S, V \setminus S)$
- induced subgraph by $S \subseteq V$ is $G(S) = (S, E(S))$
- triangles: $T(S) = \{(u, v, w) \mid (u, v), (u, w), (v, w) \in E(S)\}$

density measures

- undirected graph $G = (V, E)$
- subgraph induced by $S \subseteq V$
- clique: all vertices in S are connected to each other



density measures

- edge density (average degree):

$$d(S) = \frac{2\,|E(S,S)|}{|S|} = \frac{2\,|E(S)|}{|S|}$$

(sometimes just drop 2)

- edge ratio:

$$\delta(S) = \frac{|E(S,S)|}{\binom{|S|}{2}} = \frac{|E(S)|}{\binom{|S|}{2}} = \frac{2\,|E(S)|}{|S|(|S|-1)}$$

- triangle density:

$$t(S) = \frac{|T(S)|}{|S|}$$

- triangle ratio:

$$\tau(S) = \frac{|T(S)|}{\binom{|S|}{3}}$$

other density measures

- *k*-core: every vertex in S is connected to

  at least k other vertices in S

- α-quasiclique: the set S has at least $\alpha \binom{|S|}{2}$ edges

  i.e., S is α-quasiclique if $E(S) \geq \alpha \binom{|S|}{2}$

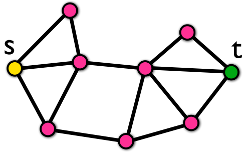
and more

not considered here

- k-cliques: subset of vertices with pairwise distances at most k
  - distances defined using intermediaries, outside the set
  - not well connected
- k-club: a subgraph of diameter $\leq k$
- k-plex: a subgraph S in which each vertex is connected to at least $|S| - k$ other vertices
  - 1-plex is a clique

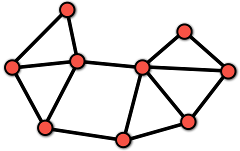
reminder: min-cut and max-cut problems

min-cut problem



- source $s \in V$, destination $t \in V$
- find $S \subseteq V$, s.t.,
- $s \in S$ and $t \in \bar{S}$, and
- minimize $e(S, \bar{S})$

max-cut problem



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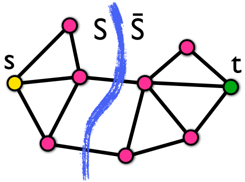
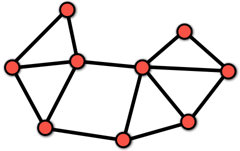


- source $s \in V$, destination $t \in V$
- find $S \subseteq V$, s.t.,
- $s \in S$ and $t \in \bar{S}$, and
- minimize $e(S, \bar{S})$
- polynomially-time solvable
- equivalent to max-flow problem

max-cut problem



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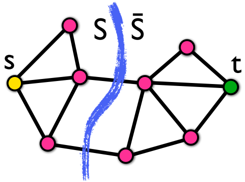

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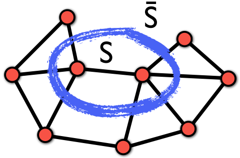
max-cut problem

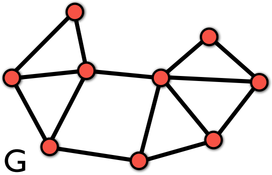


- find $S \subseteq V$, s.t.,
- maximize $e(S, \bar{S})$
- **NP**-hard
- approximation algorithms
  (0.868 based on SDP)

basic algorithms

Goldberg's algorithm for densest subgraph

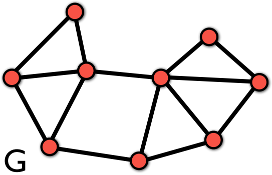
- consider first degree density d



- is there a subgraph S with $d(S) \geq c$?

Goldberg's algorithm for densest subgraph

- consider first degree density d



- is there a subgraph S with $d(S) \geq c$?
- transform to a min-cut instance

Goldberg's algorithm for densest subgraph

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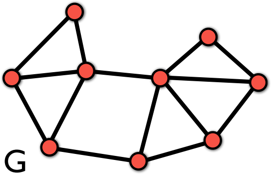

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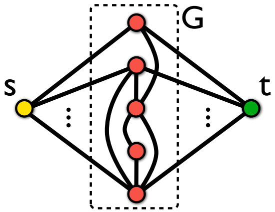


- on the transformed instance:
- is there a cut smaller than a certain value?

Goldberg's algorithm for densest subgraph

is there S with $d(S) \geq c$?

$$\frac{2\,|E(S,S)|}{|S|} \geq c$$

$$2\,|E(S,S)| \geq c|S|$$

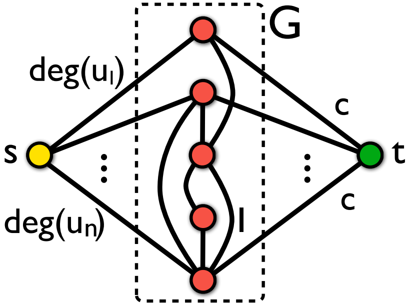
$$\sum_{u \in S} \deg(u) - |E(S,\bar{S})| \geq c|S|$$

$$\sum_{u \in S} \deg(u) + \sum_{u \in \bar{S}} \deg(u) - \sum_{u \in \bar{S}} \deg(u) - |E(S,\bar{S})| \geq c|S|$$

$$\sum_{u \in \bar{S}} \deg(u) + |E(S,\bar{S})| + c|S| \leq 2\,|E|$$

Goldberg's algorithm for densest subgraph

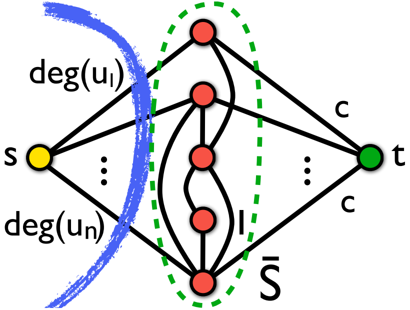
- transformation to min-cut instance



- is there S s.t. $\sum_{u \in \bar{S}} \deg(u) + |e(S, \bar{S})| + c|S| \leq 2\,|E|$?

Goldberg's algorithm for densest subgraph

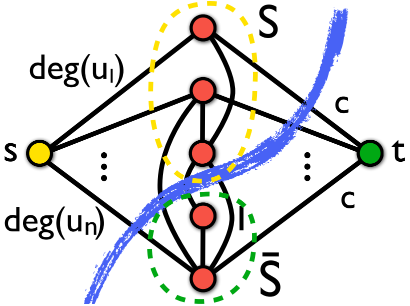
- transform to a min-cut instance



- is there S s.t. $\sum_{u \in \bar{S}} \deg(u) + |e(S, \bar{S})| + c|S| \leq 2\,|E|$?
- a cut of value $2\,|E|$ always exists, for $S = \emptyset$

Goldberg's algorithm for densest subgraph

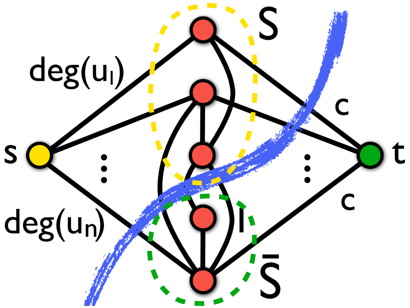
- transform to a min-cut instance



- is there S s.t. $\sum_{u \in \bar{S}} \deg(u) + |e(S, \bar{S})| + c|S| \leq 2\,|E|$?
- $S \neq \emptyset$ gives cut of value $\sum_{u \in \bar{S}} \deg(u) + |e(S, \bar{S})| + c|S|$

Goldberg's algorithm for densest subgraph

- transform to a min-cut instance



- is there S s.t. $\sum_{u \in \bar{S}} \deg(u) + |e(S, \bar{S})| + c|S| \leq 2\,|E|$?
- YES, if min cut achieved for $S \neq \emptyset$

Goldberg's algorithm for densest subgraph

[Goldberg, 1984]

input: undirected graph $G = (V, E)$, number c
output: S, if $d(S) \geq c$
1 transform G into min-cut instance $G' = (V \cup \{s\} \cup \{t\}, E', w')$
2 find min cut $\{s\} \cup S$ on G'
3 if $S \neq \emptyset$ return S
4 else return NO

- to find the densest subgraph perform binary search on c
- logarithmic number of min-cut calls
- problem can also be solved with one min-cut call
  using the parametric max-flow algorithm

densest subgraph problem – discussion

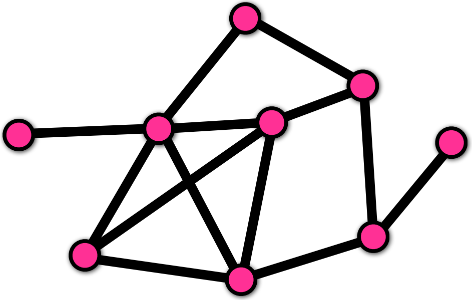
- Goldberg's algorithm polynomial algorithm, but
- $\mathcal{O}(nm)$ time for one min-cut computation
- not scalable for large graphs (millions of vertices / edges)

densest subgraph problem – discussion

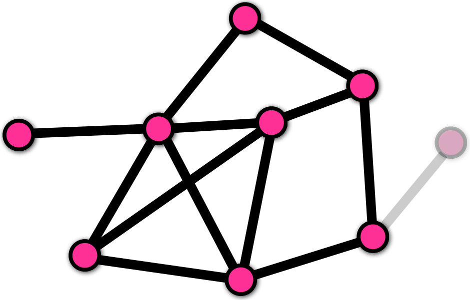
- Goldberg's algorithm polynomial algorithm, but
- $\mathcal{O}(nm)$ time for one min-cut computation
- not scalable for large graphs (millions of vertices / edges)

- faster algorithm due to [Charikar, 2000]
- greedy and simple to implement
- approximation algorithm

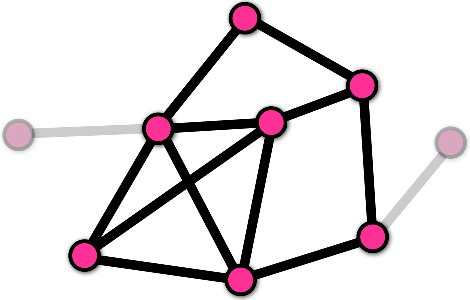
greedy algorithm for densest subgraph — example



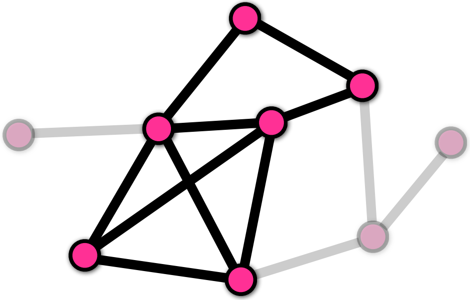
greedy algorithm for densest subgraph — example



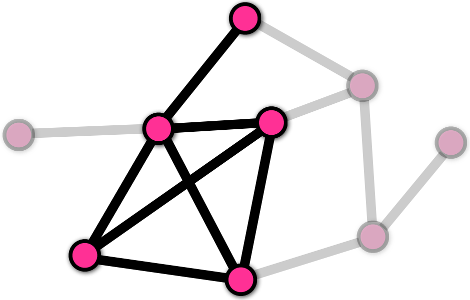
greedy algorithm for densest subgraph — example



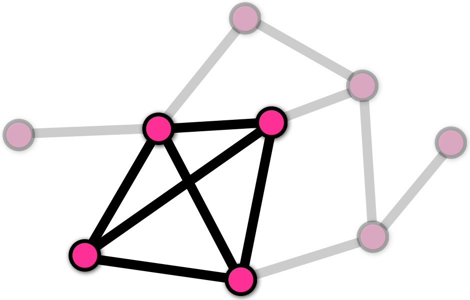
greedy algorithm for densest subgraph — example



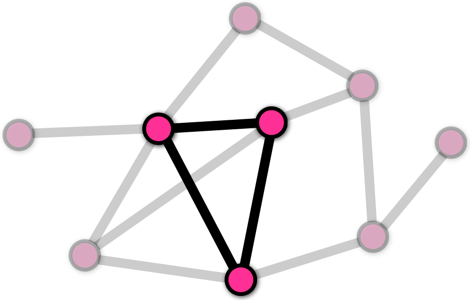
greedy algorithm for densest subgraph — example



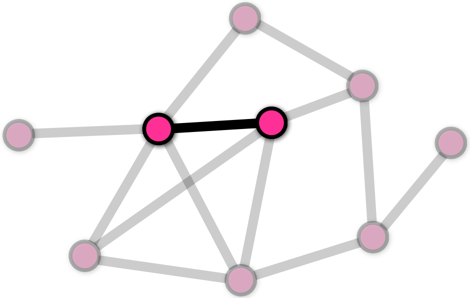
greedy algorithm for densest subgraph — example



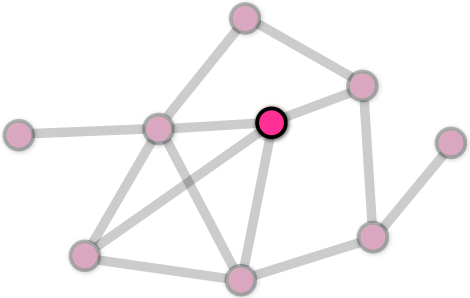
greedy algorithm for densest subgraph — example



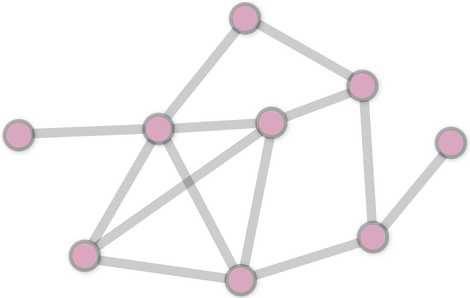
greedy algorithm for densest subgraph — example



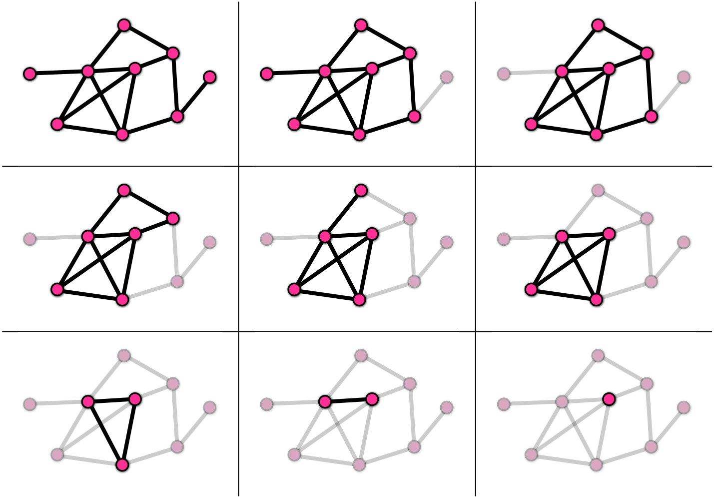
greedy algorithm for densest subgraph — example



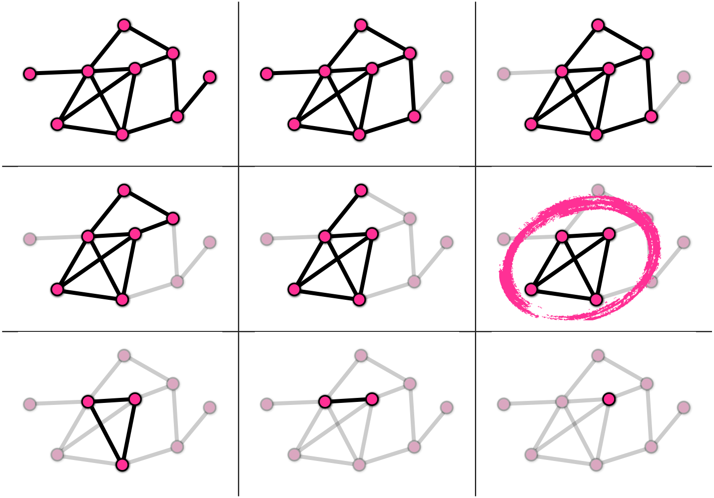
greedy algorithm for densest subgraph — example



greedy algorithm for densest subgraph — example



greedy algorithm for densest subgraph — example



greedy algorithm for densest subgraph

[Charikar, 2000]

input: undirected graph $G = (V, E)$
output: S, a dense subgraph of G

1 set $G_n \leftarrow G$
2 for $k \leftarrow n$ downto 1
2.1 let v be the smallest degree vertex in G_k
2.2 $G_{k-1} \leftarrow G_k \setminus \{v\}$
3 output the densest subgraph among $G_n, G_{n-1}, \ldots, G_1$

proof of 2-approximation guarantee

a neat argument due to [Khuller and Saha, 2009]

- let S^* be the vertices of the optimal subgraph
- let $d(S^*) = \lambda$ be the maximum degree density
- notice that for all $v \in S^*$ we have $\deg_{S^*}(v) \geq \lambda$
- (why?) by optimality of S^*

$$\frac{|e(S^*)|}{|S^*|} \geq \frac{|e(S^*)| - \deg_{S^*}(v)}{|S^*| - 1}$$

and thus

$$\deg_{S^*}(v) \geq \frac{|e(S^*)|}{|S^*|} = d(S^*) = \lambda$$

proof of 2-approximation guarantee (continued)

[Khuller and Saha, 2009]

- consider greedy when the first vertex $v \in S^* \subseteq V$ is removed
- let S be the set of vertices, just before removing v
- total number of edges before removing v is $\geq \lambda|S|/2$
- therefore, greedy returns a solution with degree density at least $\lambda/2$

QED

the greedy algorithm

- factor-2 approximation algorithm
- runs in linear time $\mathcal{O}(n + m)$
- for a polynomial problem . . .

  but faster and easier to implement than the exact algorithm

- everything works for weighted graphs

  using heaps: $\mathcal{O}(m + n \log n)$

- things are not as straightforward for directed graphs

finding dense subgraphs on directed graphs

dense subgraphs on directed graphs – history

- goal: find sets $S, T \subseteq V$ to maximize

$$d(S,T) = \frac{e[S,T]}{\sqrt{|S|\,|T|}}$$

- first introduced in unpublished manuscript [Kannan and Vinay, 1999]
- they provided a $\mathcal{O}(\log n)$-approximation algorithm
- left open the problem complexity

- polynomial-time solution using linear programming (LP) [Charikar, 2000]

dense subgraphs on directed graphs – history

[Charikar, 2000]

- exact LP-based algorithm
- greedy 2-approximation algorithm running in $\mathcal{O}(n^3 + n^2 m)$

[Khuller and Saha, 2009]

- first max-flow based exact algorithm
- improved running time of the 2-approximation greedy algorithm to $\mathcal{O}(n + m)$ (!)

directed graphs – algorithms

- reduced problem to $O(n^2)$ LP calls [Charikar, 2000]
- one LP call for each possible ratio $\frac{|S|}{|T|} = c$

$$
\begin{aligned}
\text{maximize} \quad & \sum_{(i,j)\in E(G)} x_{ij} \\
\text{such that} \quad & x_{ij} \le s_i, \quad \text{for all } (i,j) \in E(G) \\
& x_{ij} \le t_j, \quad \text{for all } (i,j) \in E(G) \\
& \sum_i s_i \le \sqrt{c} \ \text{ and } \ \sum_j t_j \le \frac{1}{\sqrt{c}} \\
& x_{ij}, s_i, t_j \ge 0
\end{aligned}
$$

directed graphs – algorithms

[Charikar, 2000]

- for a given value of $\frac{|S|}{|T|} = c$ the $\text{LP}(c)$ has an integral solution

- it can be shown that

$$\max_{S,T \subseteq V} d(S,T) = \max_{c} \text{OPT}(\text{LP}(c))$$

[proof sketch]

1. for $S, T \subseteq V$, with $\frac{|S|}{|T|} = c$ the optimal value of $\text{LP}(c)$ is at least $d(S,T)$
2. given a feasible solution of $\text{LP}(c)$ with value v we can construct $S, T \subseteq V$ such that $d(S,T) \geq v$

dense subgraphs on directed graphs – greedy

[Charikar, 2000]

input: directed graph $G = (V, E)$, ratio $c = \frac{|S|}{|T|}$

$S \leftarrow V$, $T \leftarrow V$
while both S, T non-empty
    $i_{\min} \leftarrow$ the vertex $i \in S$ that minimizes $|E(\{i\}, T)|$
    $d_S \leftarrow |E(\{i_{\min}\}, T)|$
    $j_{\min} \leftarrow$ the vertex $j \in T$ that minimizes $|E(S, \{j\})|$
    $d_T \leftarrow |E(S, \{j_{min}\})|$
    if $\sqrt{c} d_S \leq \frac{1}{\sqrt{c}} d_T$
        then $S \leftarrow S \setminus \{i_{\min}\}$
        else $T \leftarrow T \setminus \{j_{\min}\}$

- execute $\mathcal{O}(n^2)$ times; one for each $c = \frac{|S|}{|T|}$
- report best solution
- factor 2 approximation guarantee

dense subgraphs on directed graphs – greedy

- brute force execution of greedy:
  $\mathcal{O}(n^2(n+m)) = \mathcal{O}(n^3+nm))$

[Khuller and Saha, 2009]

- showed that only one execution is needed (instead of $\mathcal{O}(n^2)$)
- total running time $\mathcal{O}(n+m)$

dense subgraphs on directed graphs – greedy

linear-time greedy [Khuller and Saha, 2009]

definitions:

- let v_i, v_o be the vertices with minimum in- and out-degree
- if $d^-(v_i) \leq d^+(v_o)$ we are in category IN
  otherwise in category OUT

algorithm:

- greedy deletes the minimum-degree vertex
- if in IN, it deletes all incoming edges
- if in OUT, it deletes all outgoing edges
- if the vertex becomes a singleton, it is deleted.

- return the densest subgraph encountered

dense subgraphs on directed graphs – exact

we wish to answer "are there $S, T \subseteq V$ such that $d(S, T) \geq g$?"

consider

- consider $\alpha = \frac{|S|}{|T|}$ ($\mathcal{O}(n^2)$ possible values)
- network $G' = (\{s, t\} \cup V_1 \cup V_2, E)$, with $V_1 = V_2 = V$

min-cut transformation

- add edge of capacity m from s to each vertex of V_1 and V_2
- add edge of capacity $2m + \frac{g}{\sqrt{\alpha}}$ from each vertex of V_1 to t
- add edge from each vertex j of V_2 to sink t of capacity

$$2m + \sqrt{\alpha} g - 2\text{deg}(j)$$

- for each $(i, j) \in E(G)$, add an edge from $j \in V_2$ to $i \in V_1$ with capacity 2

dense subgraphs on directed graphs – exact

- proof of correctness of min-cut algorithm of transformed graph G' follows the argument of Goldberg
- the cut $(\{s\}, \{t, V_1, V_2\})$ has weight $m(|V_1| + |V_2|)$
- thus, min cut has weight at most $m(|V_1| + |V_2|)$
- it can be shown that solution to the min-cut with value smaller than $m(|V_1| + |V_2|)$ corresponds to sets $S \subseteq V_1$, $T \subseteq V_2$ with density $d(S, T)$ greater than g
- densest subgraph can be found with binary search on g
- one min-cut computation suffices (using parametric max-flow algorithm)

dense subgraph problem – summary

- for the degree density measure:
- exact algorithms for undirected and directed graphs
- linear-time 2-approximation achieved by greedy

- how good are these subgraphs?
- study other measures and contrast with degree density

- no control on the size of the subgraph

k-clique densest subgraphs

motivating question

- how to go beyond edge density?
- how to search for large near-cliques

- can we combine the best of both worlds, namely
  - have poly-time solvable formulation(s) which
  - …succeeds in finding large near-cliques?

- yes: the k-clique densest subgraph problem [Tsourakakis, 2015]

k-clique densest subgraph problem

Definition (k-clique density)

for any $S \subseteq V$ we define its k-clique density $\rho_k(S)$, $k \geq 2$ as $\rho_k(S) = \frac{c_k(S)}{s}$, where $c_k(S)$ is the number of k-cliques induced by S and $s = |S|$

Problem (k-clique DSP)

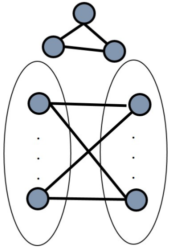
given $G(V, E)$, find a subset of vertices S^* such that $\rho_k(S^*) = \rho_k^* = \max_{S \subseteq V} \rho_k(S)$

- notice that the 2-clique DSP is simply the DSP
- we shall refer to the 3-clique DSP as the triangle densest subgraph problem

$$\max_{S \subseteq V} \tau(S) = \frac{t(S)}{s}$$

triangle densest subgraph problem

- how different can the densest subgraph be from the triangle densest subgraph?
- in principle, they can be radically different!
  consider $G = K_{n,n} \cup K_3$



- the interesting question is what happens on real-data
- can we solve the triangle DSP in polynomial time?
- an we solve the k-clique DSP in polynomial time?

triangle densest subgraph problem

Theorem

there exists an algorithm which solves the TDSP and runs in time $\mathcal{O}\left(m^{3/2} + nt + \min(n,t)^3\right)$ *where* t *is the number of triangles in the graph*

Theorem

the k*-clique DSP can be solved in polynomial time for any* $k = \Theta(1)$

- although this construction solves also the (2-clique) DSP Goldberg's algorithm is more efficient

triangle densest subgraph problem

exact algorithm

- once again, follow Goldberg's idea
- perform binary searches:

– is there a set $S \subseteq V$ such that $t(S) > \alpha |S|$?

- $\mathcal{O}(\log n)$ queries suffice to solve TDSP (why?)

– any two distinct triangle density values are at least $\mathcal{O}(1/n^2)$ away from each other

– for the optimal density $0 \leq \frac{t}{n} \leq \tau^* \leq \frac{\binom{n}{3}}{n}$

- but what does a binary search correspond to ? ...

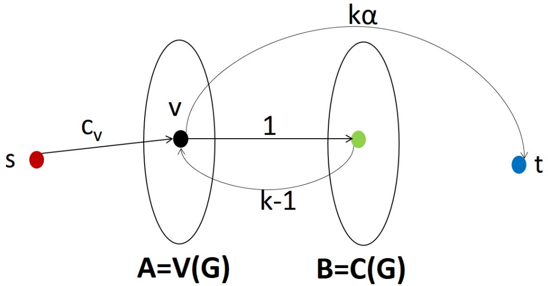
triangle densest subgraph problem

construct-network $(G, \alpha, \mathcal{T}(G))$

- $V(H) \leftarrow \{s\} \cup V(G) \cup \mathcal{T}(G) \cup \{t\}$
- for each vertex $v \in V(G)$ add an arc of capacity 1 to each triangle t_i it participates
- for each triangle $\Delta = (u, v, w) \in \mathcal{T}(G)$ add arcs to u, v, w of capacity 2
- add directed arc $(s, v) \in A(H)$ of capacity t_v for each $v \in V(G)$
- add weighted directed arc $(v, t) \in A(H)$ of capacity 3α for each $v \in V(G)$
- return network $H(V(H), A(H), w), s, t \in V(H)$

k-clique densest subgraph problem

construction for $k = \Theta(1)$



triangle densest subgraph problem

exact algorithm for TDSP

1. list the set of triangles $\mathcal{T}(G)$, $t = |\mathcal{T}(G)|$
2. $l \leftarrow \frac{t}{n}, u \leftarrow \frac{(n-1)(n-2)}{6}$
3. $S^* \leftarrow \emptyset$
4. while$(u \geq l + \frac{1}{n(n-1)})$
   - $\alpha \leftarrow \frac{l+u}{2}$
   - $H_\alpha \leftarrow$ Construct-Network$(G, \alpha, \mathcal{T}(G))$
   - $(S, T) \leftarrow$ minimum st-cut in H_α
   - if $(S = \{s\})$, then $u \leftarrow \alpha$
   - otherwise set $S^* \leftarrow \left(S \backslash \{s\}\right) \cap V(G)$ and $l \leftarrow \alpha$
5. return S^*

- run time: $\mathcal{O}\left(m^{3/2} + \left(nt + \min\left(n, t\right)^3\right) \log n\right)$
- space complexity: $\mathcal{O}(n + t)$ (typically $n \ll t$)

triangle densest subgraph problem

greedy works too

1. set $G_n \leftarrow G$
2. for $k \leftarrow n$ downto 1
   - let v be the smallest triangle count vertex in G_k
   - $G_{k-1} \leftarrow G_k \setminus \{v\}$
3. output the triangle-densest subgraph among $G_n, G_{n-1}, \ldots, G_1$

- the above peeling algorithm is a 3-approximation algorithm
- the same peeling idea generalizes to the k-clique DSP providing a k-approximation algorithm

some experimental findings

method	measure	football
DS	$\frac{\lvert S\rvert}{\lvert V\rvert}$(%)	100
	2δ	10.66
	f_e	0.094
	3τ	21.12
$\frac{1}{2}$-DS	$\frac{\lvert S\rvert}{\lvert V\rvert}$(%)	100
	2δ	10.66
	f_e	0.094
	3τ	21.12

method	measure	football
TDS	$\frac{\lvert S\rvert}{\lvert V\rvert}$(%)	15.7
	2δ	8.22
	f_e	0.48
	3τ	28
$\frac{1}{3}$-TDS	$\frac{\lvert S\rvert}{\lvert V\rvert}$(%)	15.7
	2δ	8.22
	f_e	0.48
	3τ	28

- observation 1 : approximate algorithms find the same solution as optimal exact methods
- observation 2 : the TDS is closer to being a large near-clique compared to the DS

remark

- in many cases, despite being a 2-approximation, the greedy performs optimally or close to optimally
- evidence that real-data are “far away” from adversarial

- however, 2-approximation bound is tight
- consider $G = G_1 \cup G_2$ where $G_1 = K_{d,D}$ and G_2 is the disjoint union of D cliques, each of size $d + 1$
- let $d \ll D$
- how does the greedy algorithm perform?
- optimal is bipartite clique with density $dD/(d + D) \approx d$
- greedy returns a clique of size $d + 1$ with density $d/2$

datasets

non-bipartite

dataset	n	m
Web-Google	875 713	3 852 985
Epinions	75 877	405 739
CA-Astro	18 772	198 050
Pol-blogs	1 222	16 714
Email-all	234 352	383 111

bipartite

dataset	n	m
IMDB-B	241 360	530 494
IMDB-G-B	21 258	42 197

experimental findings

k-cliques

G	$k=2$		$k=3$		$k=4$		$k=5$	
	f_e	$\|S\|$	f_e	$\|S\|$	f_e	$\|S\|$	f_e	$\|S\|$
$\star$	0.12	1 012	0.26	432	0.40	235	0.50	172
$\odot$	0.11	18 686	0.80	76	0.96	62	0.96	62
■	0.19	16 714	0.54	102	0.59	92	0.63	84
$\odot$	0.13	553	0.38	167	0.48	122	0.53	104

(p,q)-bicliques

G	$(p,q)=(1,1)$		$(p,q)=(2,2)$		$(p,q)=(3,3)$	
	f_e	$\|S\|$	f_e	$\|S\|$	f_e	$\|S\|$
$\star$	0.001	9 177	0.06	181	0.30	40
$\star$	0.001	6 437	0.41	18	0.43	17

finding densest subgraphs with map-reduce

peeling in batches

the following algorithm due to Bahmani, Kumar and Vassilvitski leads to efficient MapReduce and streaming algorithms [Bahmani et al., 2012]

1. set $S, \tilde{S} \leftarrow V$
2. **while** $S \neq \emptyset$ **do**
   - $A(S) \leftarrow \{i \in S : D_i(S) \leq 2(1+\epsilon)\rho(S)\}$
   - $S \leftarrow S \backslash A(S)$
   - **if** $\rho(S) \geq \rho(\tilde{S})$ **then** $\tilde{S} \leftarrow S$
3. return $\tilde{S}$

peeling in batches

- claim: previous algorithm is a $2(1+\epsilon)$ approximation
  furthermore, it returns after $\mathcal{O}(\log_{1+\epsilon}(n))$ rounds

- **Proof**

- approximation guarantee

– fix an optimal solution S^*

– consider the first round when a node $v \in S^*$ is removed

– let U be the set of vertices at that point

– then, $\rho^* \leq d_{S^*}(v) \leq d_U(v) \leq (2+2\epsilon)\rho(U)$

- number of rounds is $\mathcal{O}(\log_{1+\epsilon}(n))$

– in each round we throw a constant fraction of the vertices
  $2\,E(S) > \sum_{v \notin A(S)} d_S(v) > (|S| - |A(S)|)2(1+\epsilon)\rho(S)$
  and thus $|A(S)| > \frac{\epsilon}{1+\epsilon}|S|$

variations of the DSP

k-densest subgraph	$\delta(S) = \frac{2e[S]}{\lvert S\rvert}, \lvert S\rvert = k$	**NP**-hard
DalkS	$\delta(S) = \frac{2e[S]}{\lvert S\rvert}, \lvert S\rvert \geq k$	**NP**-hard
DamkS	$\delta(S) = \frac{2e[S]}{\lvert S\rvert}, \lvert S\rvert \leq k$	*L*-reduction to DkS

densest *k*-subgraph problem

- does not admit a PTAS unless $\mathbf{P} = \mathbf{NP}$
- Feige et al. gave a $\mathcal{O}(n^{\frac{1}{3}})$ approximation algorithm [Feige et al., 2001]
- state-of-the-art algorithm due to Bhaskara et al. provides a $\mathcal{O}(n^{\frac{1}{4}+\epsilon})$ approximation guarantee for any $\epsilon > 0$ [Bhaskara et al., 2010]
- closing the gap between lower and upper bounds is a significant open problem

remarks

- [Andersen and Chellapilla, 2009] proved that an α-approximation for DamkS implies a $\mathcal{O}(\alpha^2)$ approximation algorithm for the DkS
- [Khuller and Saha, 2009] improved this, by showing that an α approximation for DamkS implies a 4α approximation algorithm for the DkS
- the algorithmic ideas we showed for undirected case work for DalkS as well

an alternative density definition

edge-surplus framework

[Tsourakakis et al., 2013]

- for a set of vertices S define edge surplus

$$f(S) = g(e[S]) - h(|S|)$$

  where g and h are both strictly increasing

- optimal (g, h)-edge-surplus problem:

  find S^* such that

$$f(S^*) \geq f(S), \quad \text{for all sets } S \subseteq S^*$$

edge-surplus framework

- edge surplus $f(S) = g(e[S]) - h(|S|)$

- example 1
$$g(x) = h(x) = \log x$$
find S that maximizes $\log \frac{e[S]}{|S|}$

  densest-subgraph problem

- example 2
$$g(x) = x, \quad h(x) = \begin{cases} 0 & \text{if } x = k \\ +\infty & \text{otherwise} \end{cases}$$
k-densest-subgraph problem

the optimal quasiclique problem

- edge surplus $f(S) = g(e[S]) - h(|S|)$
- consider
$$g(x) = x, \quad h(x) = \alpha \frac{x(x-1)}{2}$$
find S that maximizes $e[S] - \alpha \binom{|S|}{2}$

  optimal quasiclique problem [Tsourakakis et al., 2013]

- theorem: let $g(x) = x$ and $h(x) = \alpha x$
  - we aim to maximize $e(S) - \alpha|S|$
  - solving $\mathcal{O}(\log n)$ such problems, solves densest subgraph problem

the edge-surplus maximization problem

theorem: let $g(x) = x$ and $h(x)$ concave

then the optimal (g, h)-edge-surplus problem is polynomially-time solvable

proof

$g(x) = x$ is supermodular

if $h(x)$ concave $h(x)$ is submodular

$-h(x)$ is supermodular

$g(x) - h(x)$ is supermodular

maximizing supermodular functions is a polynomial problem

the edge-surplus maximization problem

- poly-time solvable and interesting objectives have linear h
- the optimal quasiclique problem is **NP**-hard
- the partitioning version led to a streaming balanced graph-partitioning algorithm: FENNEL
  - goal: maximize $g(\mathcal{P})$ over all possible k-partitions where

$$g(\mathcal{P}) = \underbrace{\sum_i e[S_i, S_i]}_{\text{number of edges cut}} - \underbrace{\alpha \sum_i |S_i|^\gamma}_{\text{minimized for balanced partition}}$$

  - for more details: [Tsourakakis et al., 2014]

finding optimal quasicliques

adaptation of the greedy algorithm of [Charikar, 2000]

input: undirected graph $G = (V, E)$
output: a quasiclique S
1 set $G_n \leftarrow G$
2 for $k \leftarrow n$ downto 1
2.1 let v be the smallest degree vertex in G_k
2.2 $G_{k-1} \leftarrow G_k \setminus \{v\}$
3 output the subgraph in $G_n, \ldots, G_1$ that maximizes $f(S)$

additive approximation guarantee [Tsourakakis et al., 2013]

top-*k* dense subgraphs

top-k dense subgraphs

- in many cases we want to find more than one dense subgraph
- idea: find all dense subgraphs
  e.g., denser than a threshold
- cut enumeration techniques to output all near-optimal dense subgraphs [Saha et al., 2010]
- in practice, this method suffers from output degeneracies:
- many subsets of a dense subgraph tend to be near-optimally dense as well

top-k dense subgraphs

- another approach
  - (*i*) find a dense subgraph S
  - (*ii*) remove all vertices and edges of S
  - (*iii*) iterate
- reported subgraphs are disjoint
- certain degree of overlap can be desirable
  [Balalau et al., 2015]

top-k dense subgraphs with limited overlap

problem formulation ([Balalau et al., 2015])

- given graph $G = (V, E)$, and parameters k and α
- find k subgraphs $S_1, \ldots, S_k$
- in order to maximize

$$\sum_{i=1}^{k} d(S_i)$$

subject to

$$\frac{|S_i \cap S_j|}{|S_i \cup S_j|} \leq \alpha, \text{ for all } 1 \leq i < j \leq k$$

top-k dense subgraphs with limited overlap

algorithm MinAndRemove ([Balalau et al., 2015])

input: undirected graph $G = (V, E)$, parameters k and α
output: k subgraphs $G_1, \ldots, G_k$ with overlap at most α

while less than k subgraphs found and G non-empty
    find minimal densest subgraph $G_i = (V_i, E_i)$
    for each $v \in V_i$
        $\Delta_G(v) \leftarrow$ the set of neighbors of v in G
    remove $\lceil (1 - \alpha)|V_i| \rceil$ nodes with minimum $|\Delta_G(v) \setminus V_i|$
        and all their edges from G

top-k dense subgraphs with limited overlap

summary of results ([Balalau et al., 2015])

- MinAndRemove finds optimal solution, if this contains disjoint subgraphs
- MinAndRemove works shown to work well in practice
- faster algorithm, at small loss of accuracy

top-k dense subgraphs with limited overlap

alternative problem formulation

- given graph $G = (V, E)$, and parameters k and α
- find k subgraphs $S_1, \ldots, S_k$
- in order to maximize a reward function

$$r(S_1, \ldots, S_k) = \sum_{i=1}^{k} d(S_i) + \lambda \sum_{i,j} \mathrm{dist}(S_i, S_j)$$

- fits the max-sum diversification framework [Borodin et al., 2012]
- possible to obtain an approximation guarantee ($1/10$)

top-k dense subgraphs with limited overlap

- want to maximize

$$r(S_1, \ldots, S_k) = \sum_{i=1}^{k} d(S_i) + \lambda \sum_{i,j} \text{dist}(S_i, S_j)$$

- need to define a distance between subgraphs
- define

$$\text{dist}(S_i, S_j) = \begin{cases} 2 - \frac{|S_i \cap S_j|^2}{|S_i|\,|S_j|} & \text{if } S_i \neq S_j \\ 0 & \text{otherwise} \end{cases}$$

- distance $\text{dist}(S_i, S_j)$ is a metric function
- we can obtain an approximation guarantee ($1/10$)

top-k dense subgraphs with limited overlap

adapting the max-sum diversification framework

Algorithm 1: DOS; Algorithm for finding top-k overlapping densest subgraphs (problem Dense-Overlapping-Subgraphs)

Input: $G = (V, E), \lambda, k$
Output: set of subgraphs $\mathcal{W}$ s.t. $|\mathcal{W}| = k$ and maximizing $r(\mathcal{W})$

$\mathcal{W} \leftarrow \emptyset$;
foreach $i = 1, \ldots, k$ **do** $\mathcal{W} \leftarrow \mathcal{W} \cup \mathsf{Peel}(G, \mathcal{W}, \lambda)$;
return $\mathcal{W}$;

top-k dense subgraphs with limited overlap

adapting the max-sum diversification framework

Algorithm 2: Peel; finds a dense subgraph U of the graph G, overlapping with a collection of previously discovered subgraphs $\mathcal{W}$.

```
Input: G = (V, E), 𝒲, λ
Output: U maximizing χ(U; 𝒲)
V_n ← V;
foreach i = n, ..., 2 do
    v ← arg min_v { deg(v; V_i) − 4λ Σ_{W_j ∋ v} |V_i ∩ W_j| / |W_j| };
    V_{i−1} ← V_i \ {v};
foreach i = 1, ..., n do
    if V_i ∈ 𝒲 then V_i ← Modify(V_i, G, 𝒲, λ);
return arg max_{V_j} {χ(V_j; 𝒲)};
```

top-k dense subgraphs with limited overlap

adapting the max-sum diversification framework

Algorithm 3: Modify; modifies U if $U \in \mathcal{W}$

Input: $U, G, \mathcal{W}, \lambda$
Output: modified U

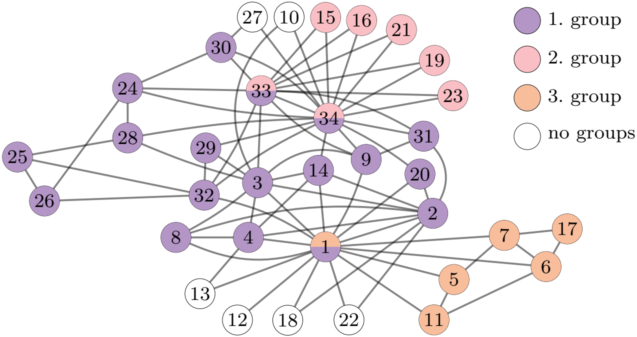
$X \leftarrow \{U \cup \{x\} \mid x \notin U,\ U \cup \{x\} \notin \mathcal{W}\}$;
$Y \leftarrow \{U \setminus \{y\} \mid y \in U,\ U \setminus \{y\} \notin \mathcal{W}\}$;
if $X = \emptyset$ **and** $\mathrm{dens}(U) \leq 5/3$ **then**
    $U \leftarrow$ {a wedge of size 3 not in $\mathcal{W}$};
else
    $U \leftarrow \arg\max_{C \in X \cup Y} \{\chi(C; \mathcal{W})\}$;
return U;

---

top-k dense subgraphs with limited overlap

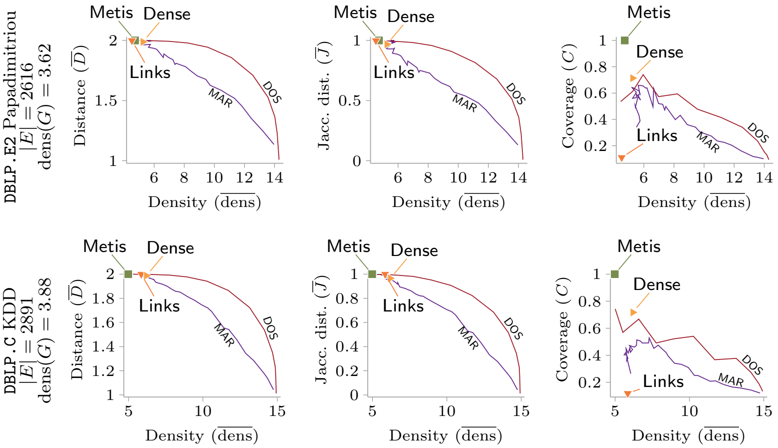
adapting the max-sum diversification framework

example



top-k dense subgraphs with limited overlap

DOS vs. MAR



core decomposition

k-core

- (recall) S is a *k*-core if every vertex in S is connected to at least k other vertices in S
- can be found with the following algorithm:
  1. while (*k*-core property is satisfied)
  2.   remove all vertices with degree less than k
- can also obtain all *k*-cores (for all k)
- all *k*-cores form a nested sequence of subgraphs (*k*-core shell decomposition)
- popular technique in social network analysis
- inner cores : more dense, more central vertices
- note resemblance with Charikar's algorithm

k-core decomposition

widely used technique for partitioning graphs

k-core $=$ largest subgraph with vertex degrees $\geq k$

cores form a chain, k-core $\subseteq (k-1)$-core; let

k-shell $=$ vertices in k-core but not in $(k+1)$-core

k-core decomposition

widely used technique for partitioning graphs

k-core $=$ largest subgraph with vertex degrees $\geq k$

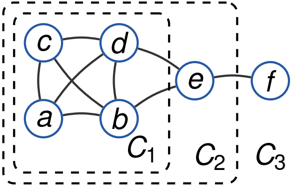
cores form a chain, k-core $\subseteq (k-1)$-core; let

k-shell $=$ vertices in k-core but not in $(k+1)$-core

algorithm to find shells:

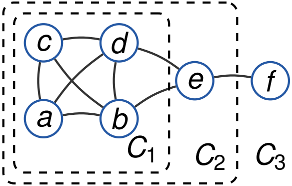
1. **while** G is not empty
2. $\quad v \leftarrow$ vertex with the smallest degree
3. $\quad$ assign v to k-shell
4. $\quad$ remove v from G

core decomposition and density are not compatible

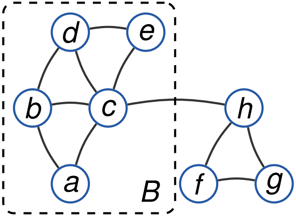


$$d(C_1) = \frac{6}{4} < \frac{8}{5} = d(C_2)$$

core decomposition and density are not compatible



$d(C_1) = \frac{6}{4} < \frac{8}{5} = d(C_2)$



only one core but

$d(B) = \frac{7}{5} > \frac{11}{8} = d(G)$

density-friendly decomposition

goal:

adapt *k*-core decomposition for density

obtain a nested sequence of increasingly dense subgraphs

[Tatti and Gionis, 2015]

locally-dense subgraphs

informally,

subgraph H is locally-dense $=$ any subgraph of H is denser than any subgraph outside H

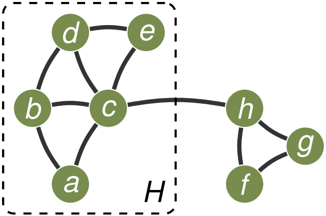
formally, define augmented density

$$d(X,Y) = \frac{|E(X)| + |E(X,Y)|}{|X|}, \quad \text{for} \quad X \cap Y = \emptyset$$

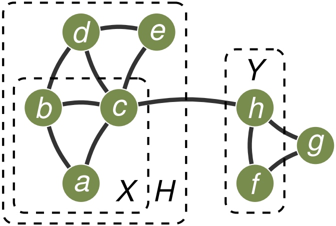
subgraph H is locally-dense if

$$d(X, H \setminus X) > d(Y, H), \quad \text{for any} \quad X \subsetneq H, Y \cap H = \emptyset$$

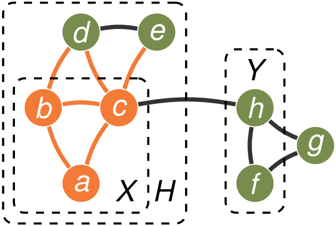
example



example

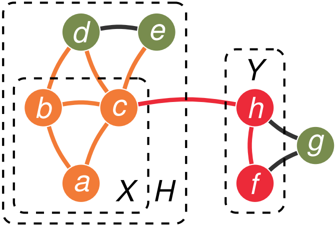


example



$$d(X, H \setminus X) = 6/3$$

example



$d(X, H \setminus X) = 6/3$
$d(Y, H) = 2/2$

properties

locally-dense subgraphs form a chain

$$\emptyset = B_0 \subsetneq B_1 \subsetneq B_2 \subsetneq \cdots \subsetneq B_k = G$$

B_i is the densest subgraph containing B_{i-1}

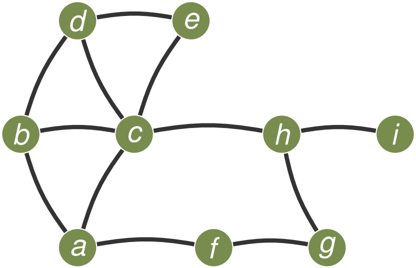
$$B_1 = \text{densest subgraph}$$

$$B_2 = \arg\max_{B \supsetneq B_1} d(B \setminus B_1, B_1)$$

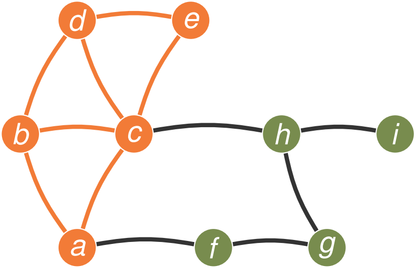
$$\cdots$$

$$B_i = \arg\max_{B \supsetneq B_{i-1}} d(B \setminus B_{i-1}, B_{i-1})$$

first approach to compute the subgraphs

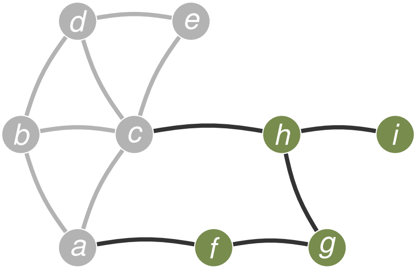


first approach to compute the subgraphs



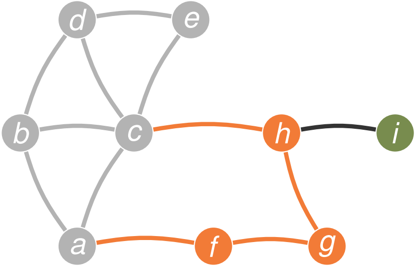
find B_1

first approach to compute the subgraphs



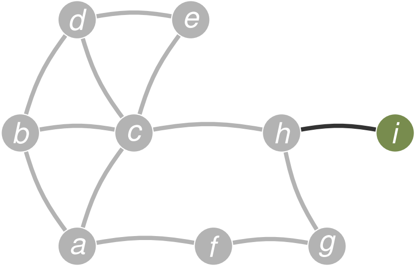
find B_1
delete B_1

first approach to compute the subgraphs



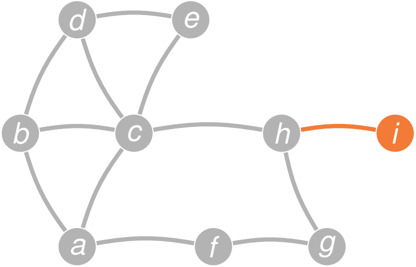
find B_1
delete B_1
find B_2

first approach to compute the subgraphs



find B_1
delete B_1
find B_2
delete B_2

first approach to compute the subgraphs



find B_1
delete B_1
find B_2
delete B_2
find B_3

computing the subgraphs

define

$$F(\alpha) = \arg\max_X |E(X)| - \alpha|X|$$

Goldberg showed that

- $F(\alpha)$ can be solved with a min-cut
- there is α such that $F(\alpha)$ is the densest subgraph

computing the subgraphs

define

$$F(\alpha) = \arg\max_X |E(X)| - \alpha|X|$$

Goldberg showed that

- $F(\alpha)$ can be solved with a min-cut
- there is α such that $F(\alpha)$ is the densest subgraph

we can show that

- $F(\alpha)$ is locally-dense
- for every B_i there is α such that $B_i = F(\alpha)$

computing the subgraphs

find all B_i by varying α (with divide-and-conquer)

algorithm: EXACT(X, Y)
1. select α such that $X \subseteq F(\alpha) \subsetneq Y$
2. $Z \leftarrow F(\alpha)$
2. **if** ($Z \neq X$)
3. **output** Z
3. EXACT(X, Z)
3. EXACT(Z, Y)

- we need only $2k - 3$ calls of $F(\alpha)$ (k is the number of locally-dense subgraphs)
- $O(n^2 m)$ total running time, in practice much faster
- $X \subset F(\alpha) \subset Y$ allows optimizations

approximation with profiles

approximation guarantees are tricky:

- algorithm may return different number of subgraphs

define a profile:

$$p(i; \mathcal{B}) = \begin{cases} d(B_1) & \text{if } i \leq |B_1| \\ d(B_2 \setminus B_1, B_1) & \text{if } |B_1| < i \leq |B_2| \\ \ldots \end{cases}$$

core decomposition

let $\mathcal{C}$ be the core decomposition

let $\mathcal{B}$ be the optimal locally-dense decomposition

then

$$p(i; \mathcal{C}) \geq p(i; \mathcal{B})/2, \quad \text{for every} \quad i$$

for $i = 1$, this implies

$$d(C_1) \geq d(B_1)/2$$

extending Charikar's algorithm

$C_1 \leftarrow$ densest subgraph of form $v_1, \ldots v_{|C_1|}$

$C_2 \leftarrow$ subgraph maximizing $d(v_1, \ldots v_{|C_2|} \setminus C_1, C_1)$

$C_3 \leftarrow$ subgraph maximizing $d(v_1, \ldots v_{|C_3|} \setminus C_2, C_2)$

$\ldots$

The graphs C_i

- can be found in $O(n^2)$-time naively
- can be found in $O(n)$-time with PAV algorithm [Ayer et al., 1955]

greedy decomposition

let $\mathcal{C}$ be the greedy decomposition
(found by the extension of Charikar's algorithm)

let $\mathcal{B}$ be the optimal locally-dense decomposition

then

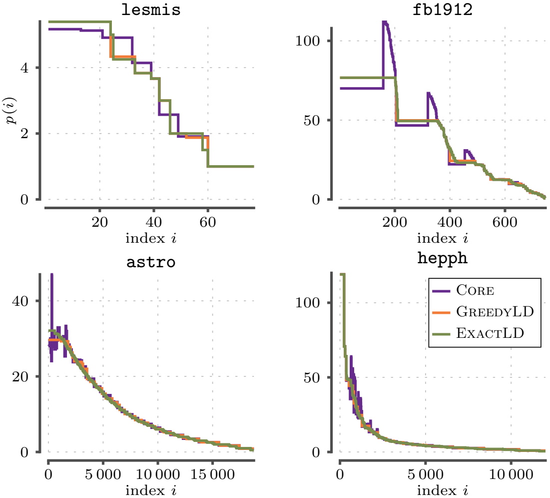
$$p(i;\mathcal{C}) \geq p(i;\mathcal{B})/2, \quad \text{for every } \; i$$

for $i = 1$, this implies

$$d(C_1) \geq d(B_1)/2$$

experiments

how well these algorithm perform?



summary (density-friendly decomposition)

- decomposition based on average density
- can be computed exactly in $\mathcal{O}(n^2 m)$ time, faster in practice
- can be $1/2$-approximated in linear time by
  - k-core decomposition
  - greedy algorithm

future work:

- consider different density functions
- control the size of the decomposition

community search

community detection problems

- typical problem formulations require non-overlapping and complete partition of the set of vertices
- quite restrictive
- inherently ambiguous: research group vs. bicycling club

- additional information can resolve ambiquity
- community defined by two or more people

the community-search problem

- given graph $G = (V, E)$, and
- given a subset of vertices $Q \subseteq V$ (the query vertices)
- find a community H that contains Q

applications

- find the community of a given set of users (cocktail party)
- recommend tags for an image (tag recommendation)
- form a team to solve a problem (team formation)

center-piece subgraph

[Tong and Faloutsos, 2006]

- given: graph $G = (V, E)$ and set of query vertices $Q \subseteq V$
- find: a connected subgraph H that
  (a) contains Q
  (b) optimizes a goodness function $g(H)$

- main concepts:
- `k_softAND`: a node in H should be well connected to at least k vertices of Q
- $r(i, j)$ goodness score of j wrt $q_i \in Q$
- $r(Q, j)$ goodness score of j wrt Q
- $g(H)$ goodness score of a candidate subgraph H
- $H^* = \arg\max_H g(H)$

center-piece subgraph

[Tong and Faloutsos, 2006]

- $r(i, j)$ goodness score of j wrt $q_i \in Q$

  probability to meet j in a random walk with restart to q_i

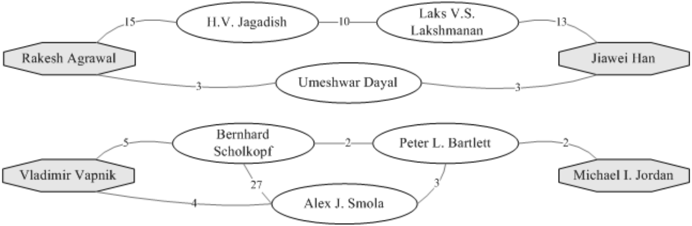
- $r(Q, j)$ goodness score of j wrt Q

  probability to meet j in a random walk with restart to k vertices of Q

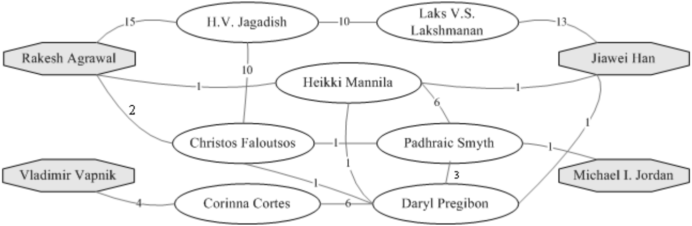
- proposed algorithm:

1. greedy: find a good destination vertex j ito add in H
2. add a path from each of top-k vertices of Q path to j
3. stop when H becomes large enough

center-piece subgraph — example results



(a) “K_softANDquery”: $k = 2$



(b) “AND query”

[Tong and Faloutsos, 2006]

the community-search problem

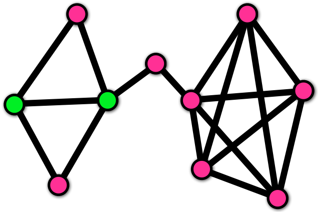
- given: graph $G = (V, E)$ and set of query vertices $Q \subseteq V$
- find: a connected subgraph H that
  (a) contains Q
  (b) optimizes a density function $d(H)$
  (c) possibly other constraints

- density function (b):

  average degree, minimum degree, quasiclique, etc.

  measured on the induced subgraph H

free riders



- remedy 1: use min degree as density function
- remedy 2: use distance constraint

$$d(Q, j) = \sum_{q \in Q} d^2(q_i, j) \leq B$$

the community-search problem

adaptation of the greedy algorithm of [Charikar, 2000]

input: undirected graph $G = (V, E)$, query vertices $Q \subseteq V$
output: connected, dense subgraph H

1 set $G_n \leftarrow G$
2 for $k \leftarrow n$ downto 1
2.1 remove all vertices violating distance constraints
2.2 let v be the smallest degree vertex in G_k among all vertices not in Q
2.3 $G_{k-1} \leftarrow G_k \setminus \{v\}$
2.4 if left only with vertices in Q or disconnected graph, stop
3 output the subgraph in $G_n, \ldots, G_1$ that maximizes $f(H)$

properties of the greedy algorithm

- returns optimal solution if no size constraints
- upper-bound constraints make the problem **NP**-hard
  (heuristic solution, also adaptation of the greedy)
- generalization for monotone constraints and
  monotone objective functions

experimental evaluation (qualitative summary)

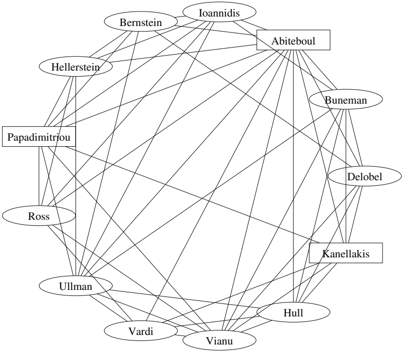
baseline: increamental addition of vertices

- start with a Steiner tree on the query vertices
- greedily add vertices
- return best solution among all solutions constructed

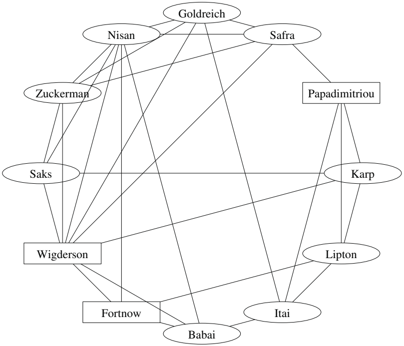
example result in DBLP

- proposed algorithm: min degree = 3, avg degree = 6
- baseline algorithm: min degree = 1.5, avg degree = 2.5

the community-search problem — example results



(a) Database theory



(b) Complexity theory

(from [Sozio and Gionis, 2010])

monotone functions

function f is monotone non-increasing if

for every graph G and

for every subgraph H of G it is

$$f(H) \leq f(G)$$

the following functions are monotone non-increasing:

- the query nodes are connected in H (0/1)
- are the nodes in H able to perform a set of tasks?
- upper-bound distance constraint
- lower-bound constraint on the size of H

generalization to monotone functions

generalized community-search problem

given

- a graph $G = (V, E)$
- a node-monotone non-increasing function f
- $f_1, \ldots, f_k$ non-increasing boolean functions

find

- a subgraph H of G
- satisfying $f_1, \ldots, f_k$ and
- maximizing f

generalized greedy

1 set $G_n \leftarrow G$
2 for $k \leftarrow n$ downto 1
2.1 remove all vertices violating any constraint $f_1, \ldots, f_k$
2.2 let v minimizing $f(G_k, v)$
2.3 $G_{k-1} \leftarrow G_k \setminus \{v\}$
3 output the subgraph H in $G_n, \ldots, G_1$ that maximizes $f(H, v)$

generalized greedy

theorem

generalized greedy computes an optimum solution
for the generalized community-search problem

running time

- depends on the time to evaluate the functions $f_1, \ldots, f_k$
- formally $\mathcal{O}(m + \sum_i nT_i)$
- where T_i is the time to evaluate f_i

heavy subgraphs

discovering heavy subgraphs

- given a graph $G = (V, E, d, w)$

  with a distance function $d : E \to \mathbb{R}$ on edges

  and weights on vertices $w : V \to \mathbb{R}$

- find a subset of vertices $S \subseteq V$

  so that

1. total weight in S is high
2. vertices in S are close to each other

[Rozenshtein et al., 2014]

discovering heavy subgraphs

- what does total weight and close to each other mean?
- total weight

$$W(S) = \sum_{v \in S} w(v)$$

- close to each other

$$D(S) = \sum_{u \in S} \sum_{v \in S} d(u, v)$$

- want to maximize $W(S)$ and minimize $D(S)$
- maximize

$$Q(S) = \lambda \, W(S) - D(S)$$

applications of discovering heavy subgraphs

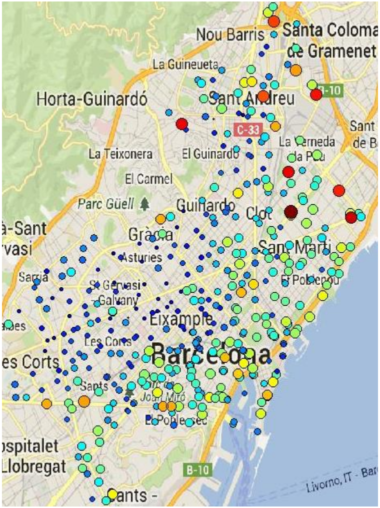
- finding events in networks
- vertices correspond to locations
- weights model activity recorded in locations
- distances between locations
- find compact regions (neighborhoods) with high activity

event detection

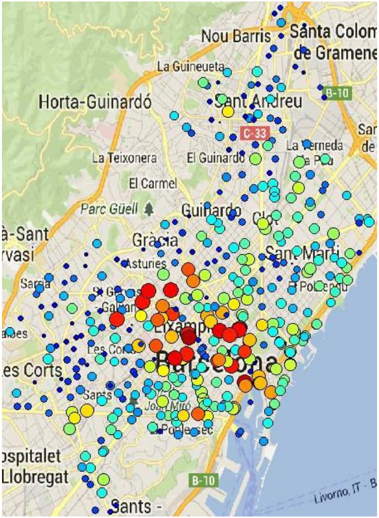
- sensor networks and traffic measurements

event detection

15.11.2012
ordinary day, no events

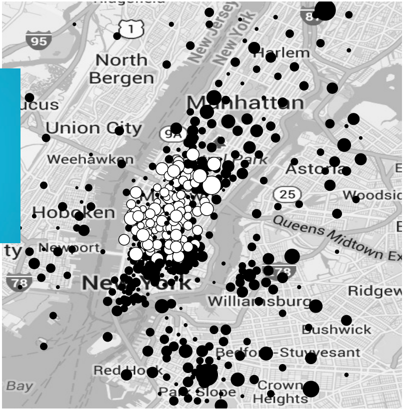


11.09.2012
Catalunya national day



event detection

- location-based social networks



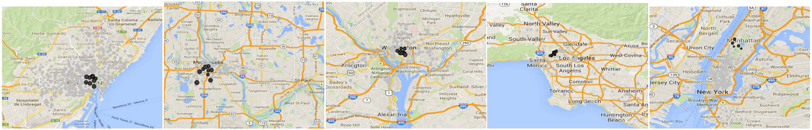
discovering heavy subgraphs

- maximize $Q(S) = \lambda\, W(S) - D(S)$
- objective can by negative
- add a constant term to ensure non-negativity
- maximize $Q(S) = \lambda\, W(S) - D(S) + D(V)$

discovering heavy subgraphs

- maximize $Q(S) = \lambda\, W(S) - D(S) + D(V)$
- objective is submodular (but not monotone)
- can obtain $\frac{1}{2}$-approximation guarantee
  [Buchbinder et al., 2012]
- problem can be mapped to the max-cut problem
  which gives 0.868-approximation guarantee
  [Rozenshtein et al., 2014]

events discovered with bicing and 4square data



(a) Barcelona: 11.09.12 National Day of Catalonia
(b) Minneapolis: 4.07.12 Independence Day
(c) Washington, DC: 27.05.13 Memorial Day
(d) Los Angeles: 31.05.10 Memorial Day
(e) New York: 6.09.10 Labor Day

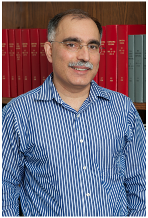
Figure 4: Public holiday city-events discovered using the SDP algorithm.

(a) 01.06.12 Primavera sound music festival
(b) 18.09.12 festival of the Poblenou neighborhood
(c) 31.10.12 Halloween

summary

- the problem of finding dense subgraphs has many different real-world applications
- a number of density measures have been studied
- problem complexity depends on adopted measure
- for some problem formulations there are exact polynomial and faster approximate solution
- a number of different techniques has been used
  min-cut, greedy, submodularity optimization
- many directions and open problems for future work

acknowledgements



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