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Online Entropy Estimator

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Outline

Introduction

Estimator for Unknown Entropy Sources

New Estimator

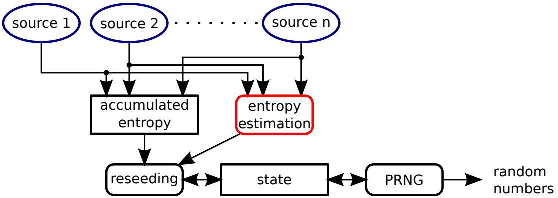
Empirical Results

Conclusion

Introduction

PRNG with Entropy Input

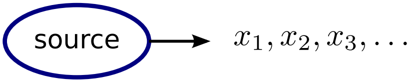
- **Pseudo Random Number Generator with entropy sources**



- **PRNG is deterministic**: A specific state will always produce the same output
- Uncertainty come from **entropy sources**
- When **"enough uncertainty is collected"** reseed the state
  - **How do we know when to reseed?**

Entropy - Information Theoretical Model

- **Entropy Source**



- Outputs element from **sample space** $\mathcal{X}$
- **Concrete output** at **time** t: $x_t \in \mathcal{X}$
- Source represents sequence of identical and independent distributed **random variables** $X_1, X_2, \ldots$

$$\Pr(X_t = \eta) = p_\eta \text{ for all } t,\ p = (p_\eta)_{\eta \in \mathcal{X}}$$

- **Sequence** of **random variables** $\mathbf{X}_{[t_1,t_2]} = X_{t_1}, X_{t_1+1}, \ldots, X_{t_2}$
- **Sequence** of **concrete outputs** $\mathbf{x}_{[t_1,t_2]} = x_{t_1}, x_{t_1+1}, \ldots, x_{t_2}$
- **Empirical distribution** of $\mathbf{x}_{[1,n]}$: $\hat{p}_\eta = \frac{\#\{t : x_t = \eta\}}{n}$

Entropy - Definitions

- **Shannon entropy**: Measure of average number of binary questions before guessing the output
$$H(p) = -\sum_{\eta \in \mathcal{X}} p_\eta \log_2 p_\eta$$
- **Rényi entropy**: Measure of correlation probability
$$H_\alpha(p) = \frac{1}{1-\alpha} \log_2 \left(\sum_{\eta \in \mathcal{X}} p_\eta^\alpha \right)$$
- **Min entropy**: Lower bound for all entropy measures
$$H_\infty(p) = -\log_2 \left(\max_{\eta \in \mathcal{X}} p_\eta \right)$$
- **Relation** for $\alpha > 1$:
$$H_\infty(p) \leq H_\alpha(p) \leq H(p) \leq \log_2 |\mathcal{X}|$$

Estimator for Unknown Entropy Sources

Requirements

- Estimator $\hat{H}$ should:
  - Work with **unknown sources**
  - Be **pessimistic** $E(\hat{H}) \leq H(p)$
  - Be **efficient**
  - Given an estimate for **each output**
    - Want $\hat{H}$ such that $\frac{1}{n-r} \sum_{t=r+1}^{n} \hat{H}(\mathbf{X}_{[t-r,t]}) \xrightarrow{n \to \infty} H(p)$
  - Work with **any source**

Known Estimators (1)

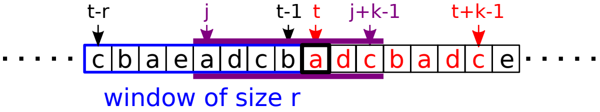
- **Plug-in** or **maximum-likelihood** estimator: use $H(\hat{p})$
  - Estimate only for **whole data set**
  - Need counter for **each** $\eta \in \mathcal{X}$
  - Can be applied on a data window but still not efficient

- **Compression** based on **frequency counting**:
  e.g. Huffman coding, arithmetic coding
  - **Not efficient** (Huffman coding need tree of size $|\mathcal{X}|$)

Known Estimators (2)

- **Compression** based on **match length** [Lempel Ziv]

$$L_t^r(\mathbf{x}) = 1 + \max_{t-r \leq j \leq t-1} \{k : \mathbf{x}_{[t,t+k-1]} = \mathbf{x}_{[j,j+k-1]}\}$$



- **Classical estimator**:

$$\frac{L_t^r(\mathbf{X})}{\log_2 r} \to \frac{1}{H} \text{ a.s. } (r \to \infty)$$

  - Estimate only for **whole data set**
  - Need **future values**, no upper bound for $L_t^r(\mathbf{x})$

Known Estimators (3)

- **LZ estimator** with intermediate values, $\hat{H}^r_{\mathrm{LZ}}(\mathbf{x}_{[t,n]}) = \frac{\log_2 r}{L^r_t(\mathbf{x})}$

$$\frac{1}{n-r} \sum_{t=r+1}^{n} \hat{H}^r_{\mathrm{LZ}}(\mathbf{x}_{[t,n]}) \to H \text{ a.s. } (n, r \to \infty)$$

  - Gives an estimate for **each** $t \geq r$
  - Need **future values**, no upper bound for $L^r_t(\mathbf{x})$

Known Estimators (4)

- Based on **transition frequencies** [Bucci Luzzi 2005]
- **Count transitions** from 0 to 1 or from 1 to 0
- **Expected number of transition** in $n+1$ bits: $n2p_0(1-p_0)$
- Use: $-log_2 y \geq \frac{1}{\ln(2)}(1-y)$ for $0 < y < 1$
- **Entropy**:

$$-p_0 \log_2 p_0 - (1-p_0)\log_2(1-p_0) \geq \frac{1}{\ln(2)} 2p_0(1-p_0)$$

  - Only **binary sources**

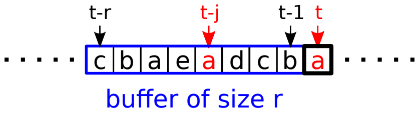
- **Our idea**: Extend to **non-binary case**

New Estimator

Idea

- **Number of comparisons** before finding **last occurrence** of **current element**:

$$\ell_t^r = \begin{cases} r & \text{if } x_t \neq x_{t-j}, 1 \leq j \leq r, \\ \min\{0 \leq j \leq r-1 : x_{t-1-t} = x_t\} & \text{otherwise.} \end{cases}$$



- **Estimator:** $\hat{H}^r_{\mathrm{pv}}(\mathbf{x}_{[t-r,t]}) = \frac{1}{\ln(2)} \sum_{j=1}^{\ell_t^r} \frac{1}{j}$

Theory

- **Expected value**: $\mathrm{E}\left(\hat{H}^r_{\mathrm{pv}}(\mathbf{X}_{[t-r,t]})\right) = \frac{1}{\ln(2)} \sum_{\eta\in\mathcal{X}} p_\eta \sum_{i=1}^{r} \frac{(1-p_\eta)^i}{i}$
- Using results on **(r+1)-dependent random variables**:

$$\frac{1}{n-r}\sum_{t=r+1}^{n} \hat{H}^r_{\mathrm{pv}}(\mathbf{x}_{[t,n]}) \to \mathrm{E}\left(\hat{H}^r_{\mathrm{pv}}(\mathbf{X}_{[t-r,t]})\right) \text{ a.s. } (n\to\infty)$$

- Taylor series of logarithm ($0 < x < 1$):

$$\ln\frac{1}{x} = \sum_{i=1}^{\infty}\frac{(1-x)^i}{i} \geq \sum_{i=1}^{r}\frac{(1-x)^i}{i}$$

- **Lower bound for entropy**:

$$\mathrm{E}\left(\hat{H}^r_{\mathrm{pv}}(\mathbf{X}_{[t-r,t]})\right) \leq \sum_{\eta\in\mathcal{X}} p_\eta \log_2\frac{1}{p_\eta} = H(p)$$

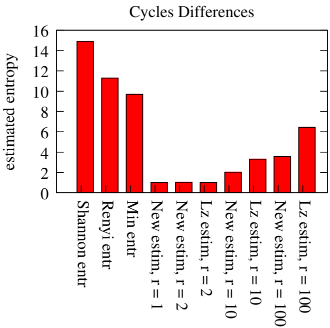
Empirical Results

Test Data

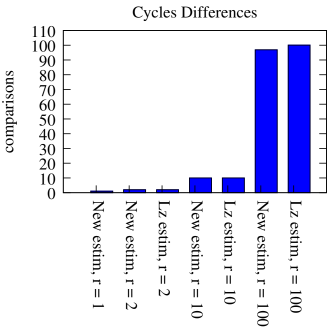
- Input data for **Linux random number generator**
  - Cooperation with Lacharme, Strubel, Videau
  - Time between interrupts from **user input (mouse, keyboard)**
  - Time measured in **jiffies** and **cycles**
- Data from **iPhone GPS** device
  - Cooperation with Lauradoux, Ponge
  - Measurement of altitude, longitude, latitude, acceleration and compass (heading)
  - **Indoor** (less movement) and **outdoor** (more movement) measurements
- Compare to **LZ estimator**

Linux RNG - Cycle Count Difference

Entropy



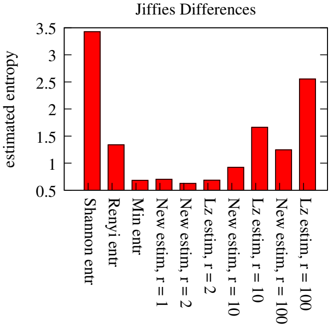
Number of comparisons



- **LZ estimator** has **complexity as good as our** estimator

Linux RNG - Jiffies Count Difference

Entropy



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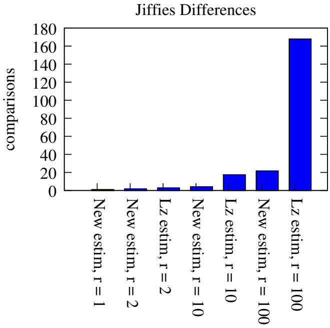
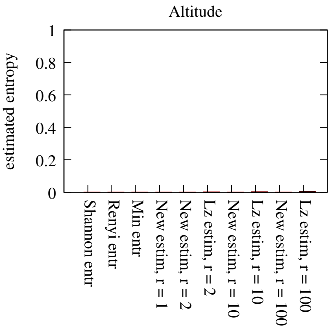


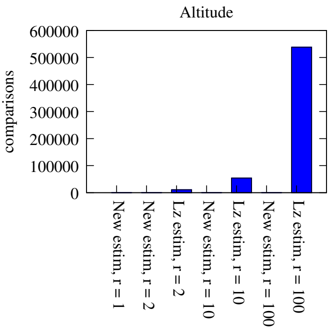
- **Better complexity** than LZ estimator

iPhone - Altitude - Indoor

Entropy



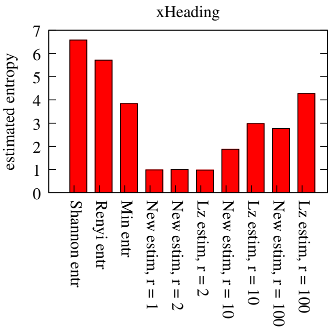
Number of comparisons



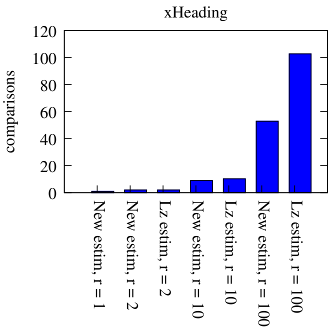
- **Extreme case for LZ estimator** if source has (almost) **no entropy**

iPhone - xHeading - Outdoor

Entropy



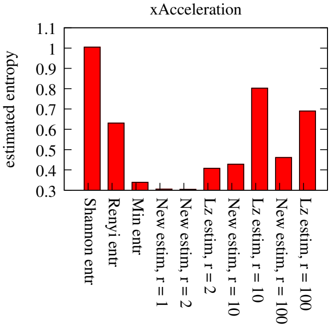
Number of comparisons



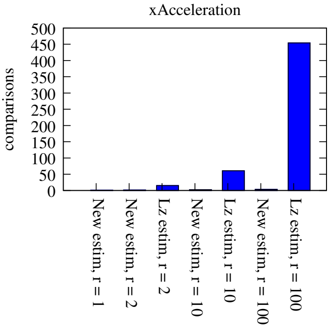
- Estimated entropy of new estimator **not far from LZ estimate**

iPhone - xAcceleration - Indoor

Entropy



Number of comparisons



- **Better complexity** than LZ estimator

Conclusion

Conclusion

- **Entropy estimator** for **unknown** entropy sources with **changing behavior**
- **At most r comparisons**
- For **independent sequences** the **expected value** of the estimator is **lower bound** for entropy
- Gives estimate for **each output value**

- **Only Shannon** entropy, not Rényi or Min entropy
- Proof **not** for **correlated sources**
- Estimates **lower bound**, exact value only for $r \to \infty$