Brief Announcement:
Linial’s Lower Bound Made Easy

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Problem: 3-colouring cycles
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- Directed cycle with $n$ nodes
- $O(\log n)$-bit identifiers, LOCAL model
Problem: 3-colouring cycles

- Linial (1992): requires at least $\frac{1}{2} \log^*(n) - 1$ communication rounds

- Today: same result, simpler proof
  - student-friendly, self-contained, < 2 pages
  - no references to neighbourhood graphs, line graphs, chromatic numbers …
Colouring cycles in time $T$

- Each node must output its own colour
- Running time $T = \text{output only depends on radius-}$T neighbourhood of the node
Colouring cycles in time $T$

$A(87, 29, 11, 46, 32) \neq A(29, 11, 46, 32, 77)$

$T = 2$
$k$-ary $c$-colouring function

$A(25, 29, 34, 46, 52) \neq A(29, 34, 46, 52, 77)$

$k = 5$

$A(25, 29, 34, 46, 52)$

$A(29, 34, 46, 52, 77)$
$k$-ary $c$-colouring function

- $A(x_1, x_2, \ldots, x_k) \in \{1, 2, \ldots, c\}$ for all $1 \leq x_1 < x_2 < \ldots < x_k \leq n$

- $A(x_1, x_2, \ldots, x_k) \neq A(x_2, x_3, \ldots, x_{k+1})$ for all $1 \leq x_1 < x_2 < \ldots < x_{k+1} \leq n$
**k-ary c-colouring function**

- Assume: $A$ is a distributed algorithm that finds a 3-colouring in directed $n$-cycles in time $T$

- Then: $A$ is a *k-ary 3-colouring function* for $k = 2T + 1$

- Plan: show that $k + 1 \geq \log^* n$
Lemma 1

- If there is a 1-ary $c$-colouring function, then $c \geq n$

- Proof:
  - pigeonhole principle
Lemma 2

• Given: a \( k \)-ary \( c \)-colouring function \( A \)

• We can construct:
  a \((k - 1)\)-ary \( 2^c \)-colouring function \( B \)

• Proof:
  \[
  B(x_1, x_2, \ldots, x_{k-1}) = \{A(x_1, x_2, \ldots, x_{k-1}, y) : y > x_{k-1}\}
  \]
Iterate Lemma 2

- $k$-ary 3-colouring function $\rightarrow$
  - $k$-ary $2^2$-colouring function $\rightarrow$
  - $(k - 1)$-ary $3^2$-colouring function $\rightarrow$
  - $(k - 2)$-ary $4^2$-colouring function $\rightarrow$
  - ...
- 1-ary $k+1^2$-colouring function

$i2 = 2^{i\cdot2}$ (i twos)
Conclusion

• **Lemma 2:**
  - \( k \)-ary 3-colouring function \( \rightarrow \) 1-ary \( k+1 \)-colouring function

• **Lemma 1:**
  - \( k+1 \) \( \geq \) \( n \) (that is, \( k + 1 \geq \log^* n \))

\[ i2 = 2^2 \cdots 2 \quad (i \text{ twos}) \]