Towards a complexity theory for the congested clique

Janne H. Korhonen · janne.h.korhonen@aalto.fi
Aalto University

Jukka Suomela · jukka.suomela@aalto.fi
Aalto University

Abstract. The congested clique model of distributed computing has been receiv-
ing attention as a model for densely connected distributed systems. While there
has been significant progress on the side of upper bounds, we have very little
in terms of lower bounds for the congested clique; indeed, it is now know that
proving explicit congested clique lower bounds is as difficult as proving circuit
lower bounds.

In this work, we use various more traditional complexity-theoretic tools to
build a clearer picture of the complexity landscape of the congested clique:

– Nondeterminism and beyond: We introduce the nondeterministic congested
clique model (analogous to NP) and show that there is a natural canonical
problem family that captures all problems solvable in constant time with
nondeterministic algorithms. We further generalise these notions by intro-
ducing the constant-round decision hierarchy (analogous to the polynomial
class hierarchy).

– Non-constructive lower bounds: We lift the prior non-uniform counting
arguments to a general technique for proving non-constructive uniform lower
bounds for the congested clique. In particular, we prove a time hierarchy
theorem for the congested clique, showing that there are decision problems of
essentially all complexities, both in the deterministic and nondeterministic
settings.

– Fine-grained complexity: We map out relationships between various natural
problems in the congested clique model.
1 Introduction

The congested clique. In this work, we study computational complexity questions in the congested clique model of distributed computing. The congested clique is essentially a fully-connected specialisation of the classic CONGEST model of distributed computing: There are $n$ nodes that communicate with each other in a fully-connected synchronous network by exchanging messages of size $O(\log n)$. Each node in the network corresponds to a node in an input graph $G$, each node starts with knowledge about their incident edges in $G$, and the task is to solve a graph problem related to $G$.

The congested clique has recently been receiving increasing attention especially on the side of the upper bounds, as it has been discovered that the fully-connected network topology allows for significantly faster algorithms than what is possible in the CONGEST model. However, on the side of complexity theory, there has been significantly less development. Compared to the LOCAL and CONGEST models, where complexity-theoretic results have generally taken the form of unconditional, explicit lower bounds for concrete problems, such developments have not been forthcoming in the congested clique. Indeed, it was show by Drucker et al. [13] that congested clique lower bounds imply circuit lower bounds, and the latter are notoriously difficult to prove – overall, it seems that there are many parallels between computational complexity in the congested clique and centralised computational complexity.

Towards a complexity theory. We use concepts and techniques from centralised complexity theory to map out the complexity landscape of the congested clique model. First, we focus on decision problems:

- We introduce the nondeterministic version of the congested clique model. In particular, the class $\text{NCLIQUE}(O(1))$ of problems solvable in constant time with nondeterministic algorithms is a natural analogue of the class NP. We show that there is a natural canonical problem family that captures all $\text{NCLIQUE}(O(1))$ problems.

- We further generalise the notion of nondeterministic congested clique by introducing the constant-round decision hierarchy, analogous to the polynomial hierarchy.

- We prove time hierarchy theorems for the congested clique, showing that there are decision problems of essentially all complexities both in deterministic and nondeterministic settings.

Furthermore, we study the landscape of natural graph problems in the congested clique using a fine-grained complexity approach:

- While we cannot prove explicit lower bounds for the congested clique, we map out the relative complexity of problems with polynomial complexity.

1.1 Results: time hierarchy

It is known that in the centralised setting, there are problems of almost any deterministic time complexity, due to the time hierarchy theorem [21, 25]. However, in distributed computing, we know that the picture can be quite different; for LCL problems in the LOCAL model, there are known complexity gaps, implying that LCL problems can have only very specific complexities [5, 8, 39]. Thus, it makes sense to ask how the picture looks like in the congested clique: for example, it could be that – similarly to LCL problems in the LOCAL model – there are no problems with complexity $o(\log^* n)$ and $\omega(1)$. 
We show that no such gaps occur in the congested clique model. Writing $\text{CLIQUE}(O(T(n)))$ for the set of decision problems that can be solved in $O(T(n))$ rounds, we prove a time hierarchy theorem for the congested clique: for any sensible complexity functions $S$ and $T$ with $S(n) = o(T(n))$, we have that

$$\text{CLIQUE}(O(S(n))) \subsetneq \text{CLIQUE}(O(T(n))).$$

The proof of the time hierarchy theorem is based on the earlier circuit counting arguments for a non-uniform version of the congested clique [1, 13]. We show how to lift this result into the uniform setting, allowing us to show the existence of decision problems of essentially arbitrary complexity. Indeed, we use this same technique also for the other separation results in this paper.

1.2 Results: nondeterminism and beyond

**Nondeterministic congested clique.** The class NP and NP-complete problems are central in our understanding of centralised complexity theory. We build towards a similar theory for the congested clique by introducing a *nondeterministic congested clique model*. We define the class $\text{NCLIQUE}(O(T(n)))$ as the class of decision problems that have nondeterministic algorithms with running time $O(T(n))$, or equivalently, as the set of decision problems $L$ for which there exists a deterministic algorithm $A$ that runs in $O(T(n))$ rounds and satisfies

$$G \in L \text{ if and only if } \exists z: A(G, z) = 1,$$

where $z$ is a *labelling* assigning each node $v$ a nondeterministic guess $z_v$; for details, see Section 5.

We show that nondeterminism is only useful up to the number of bits communicated by the algorithm: any nondeterministic algorithm with running time $O(T(n))$ can be converted to a *normal form* where each yes-instance has an accepting labelling with $|z_v| \leq O(T(n)n \log n)$. As an application of this result, we show that $\text{NCLIQUE}(O(S(n)))$ is not contained in $\text{CLIQUE}(O(T(n)))$ for any $S(n) = o(T(n))$.

**Constant-round nondeterministic decision.** We argue that the class $\text{NCLIQUE}(O(1))$ of problems solvable in constant time with nondeterministic algorithms is a natural analogue of the class NP. The class $\text{NCLIQUE}(O(1))$ contains most natural decision problems that have been studied in the congested clique, as well as many NP-complete problems such as $k$-colouring and Hamiltonian path. In particular, the question of proving that

$$\text{CLIQUE}(O(1)) \neq \text{NCLIQUE}(O(1))$$

can be seen as playing a role similar to the P vs. NP question in the centralised setting. Alternatively, $\text{NCLIQUE}(O(1))$ can be seen as an analogue of the class LCL of locally checkable labellings that has been studied extensively in the context of the LOCAL model.

While we cannot prove a separation between deterministic and nondeterministic constant time, we identify a family of *canonical problems* for $\text{NCLIQUE}(O(1))$: we show that any $\text{NCLIQUE}(O(1))$ problem can be formulated as a specific type of *edge labelling problem*. In particular, showing a non-constant lower bound for any edge labelling problem would be sufficient to separate $\text{CLIQUE}(O(1))$ and $\text{NCLIQUE}(O(1))$.

**Constant-round decision hierarchy.** We extend the notion of nondeterministic clique by studying a *constant-round decision hierarchy*. This can be seen as analogous to the polynomial
hierarchy in the centralised setting; each node can be seen as running an alternating Turing machine.

Unlike for nondeterministic algorithms, it turns out that the label size for algorithms on the higher levels of this hierarchy is not bounded by the amount of communication. Thus, we get two very different versions of this hierarchy:

- **Unlimited hierarchy** \((\Sigma_k, \Pi_k)_{k=1}^{\infty}\) with unlimited label size: we show that this version of the hierarchy collapses, as all decision problems are contained on the second level.

- **Logarithmic hierarchy** \((\Sigma_{k}^{\log}, \Pi_{k}^{\log})_{k=1}^{\infty}\) with label size \(O(n \log n)\) per node: we show that there are problems that are not contained in this hierarchy.

### 1.3 Results: fine-grained complexity

By the time hierarchy theorem, we know that there are decision problems of all complexities, but it is beyond our current techniques to prove lower bounds for any specific problem, assuming we exclude lower bounds resulting from input or output sizes. However, what we can do is study the relative complexity of natural problems, much in the vein of centralised fine-grained complexity. In Section 7, we study the relative complexities of various concrete problems that are thought to have polynomial complexity in the congested clique.

### 2 Related work

**Upper bounds for the congested clique.** As noted in the introduction, upper bounds have been extensively studied in the congested clique model. Problems studied in prior work include routing and sorting [34], minimum spanning trees [17, 24, 26, 36], subgraph detection [6, 11], shortest path problems [3, 6], local problems [7, 22, 23] and problems related to matrix multiplication [6, 33].

**Complexity theory for the congested clique.** Prior work on computational complexity in the congested clique is fairly limited; the notable exceptions are the connections to circuit complexity [13] and counting arguments for the non-uniform version of the model [1, 13]. However, lower bounds can be proven if we consider problems with large outputs; for example, lower bounds are known for triangle enumeration [40] or, trivially, a problem where all nodes are required to output the whole input graph. Moreover, for the broadcast congested clique, a version of the model where each node sends the same message to each other node every round, lower bounds can be proven using communication complexity arguments [13].

**Complexity theory for CONGEST.** For the CONGEST model, explicit lower bounds are known for many problems, even on graphs with very small diameter [10, 16, 31, 35, 38, 41]. These are generally based on reductions from known lower bounds in communication complexity; however, these reductions tend to boil down to constructing graphs with bottlenecks, that is, graphs where large amounts of information have to be transmitted over a small cut. A key motivation for the study of the congested clique model is to understand computation in networks that do not have such bottlenecks.

**Complexity theory for LOCAL.** Perhaps the most active development related to the computational complexity theory of distributed computing is currently taking place in the context of the LOCAL model. There is a lot of very recent work that aims at developing a complete classification of the complexities of LCL problems in the LOCAL model [4, 5, 8, 9, 18]. In this line of research, the focus is on low-degree large-diameter graphs, while in the congested
clique model we will study the opposite corner of the distributed computing landscape: high-degree low-diameter graphs.

**Nondeterminism and alternation.** Nondeterministic models of distributed computing have been studied under various names – for example, *proof labeling schemes* [27–30], *nondeterministic local decision* [15], and *locally checkable proofs* [20] can be interpreted as nondeterministic versions of some variants of the LOCAL and CONGEST models. However, there seem to be very few papers that take the next step from nondeterministic machines to alternating machines in the context of distributed computing – we are only aware of Reiter [42], who studies alternating quantifiers in finite state machines, and Feuilloley et al. [14] and Balliu et al. [2], who study alternating quantifiers in the LOCAL model.

### 3 Preliminaries

**The congested clique.** The congested clique is a specialisation of the standard CONGEST model of distributed computing to a fully connected network topology. The network consists of *n* nodes (i.e. computers) that are connected to all other nodes by edges (i.e. communication links) – that is, the communication graph is a clique.

As an input, we are given an undirected, unweighted graph *G* = (*V*, *E*) with *V* = {1, 2, ..., *n*}. Each node of the communication network has a unique identifier *v* ∈ {1, 2, ..., *n*} and, in addition to its own unique identifier, has initially knowledge about edges incident to node *v* in *G*. The nodes collaborate to solve a problem related to the graph *G*.

The computation is done in synchronous rounds, and all nodes run the same deterministic algorithm. Each round, all nodes (1) perform an unlimited amount of local communication, (2) send a possibly different *O*(log *n*)-bit message to each other node, and (3) receive the messages sent to them. The time complexity of an algorithm is measured in the number of rounds used.

**Decision problems.** To avoid artefacts resulting from input and output sizes, we restrict our attention for the most part to decision problems on unweighted, undirected graphs. A *decision problem* *L* is a family of graphs; a graph *G* is a yes-instance of *L* if *G* ∈ *L* and a no-instance otherwise. The *complement* ¯*L* of problem *L* contains all graphs *G* that are not in *L*.

Note that we do not require decision problems to be closed under isomorphisms, that is, problems can refer to the names of the nodes. However, we are only interested in decision problems that are computable in the centralised sense, and we will implicitly assume that this is the case for any problem considered.

An algorithm *A* solves problem *L* if for any graph *G*, each node *i* produces output *A* (*i* (*G*)) = 1 if *G* ∈ *L* and *A* (*i* (*G*)) = 0 if *G* / ∈ *L*; we write *A* (*G*) = 1 to indicate that all nodes produce output 1 (the algorithm *accepts*) and *A* (*G*) = 0 to indicate that all nodes output 0 (the algorithm *rejects*).

**Input encoding.** We will tacitly assume that the input is provided for node *v* ∈ *V* in the form of a length-(*n* − 1) bit vector *x* *v* indexed by *V* \ {v} describing whether each of the potential incident edges is present in the input graph *G* = (*V*, *E*). In particular, any two nodes *u* and *v* will share the bit *x* *u*,*v* = *x* *v*,*u*.

However, for technical reasons, it is convenient to consider a setting where each node has *private* input bits, so we will implicitly assume that each of these bits is assigned to only exactly one node, so that (1) for each possible edge in the graph, exactly one of its endpoints
has the bit corresponding to that edge and (2) each node has at least \(\lfloor (n - 1)/2 \rfloor\) input bits. Note that it takes a single round to move from the latter setting to the former.

**Algorithms.** As noted above, we assume all nodes run the same deterministic algorithm. While the congested clique allows for \(O(\log n)\) bandwidth per round, where the constant hidden by \(O\)-notation can depend on the algorithm, we can always move the constant factors to the running time and assume that all algorithms use exactly \(\lceil \log_2 n \rceil \) bits of communication per round.

**Deterministic complexity classes.** For a computable function \(T: \mathbb{N} \to \mathbb{N}\), we define the complexity class \(\text{CLIQUE}(T(n))\) as the family of all graph problems that can be solved in \(T(n)\) rounds. Moreover, for a family of functions \(C\), we define \(\text{CLIQUE}(C) = \bigcup_{T \in C} \text{CLIQUE}(T(n))\), where \(T\) ranges over computable functions in \(C\).

**Counting arguments.** We will now review known results on the non-uniform version of the congested clique [1, 13]. Specifically, we consider a setting where the number of nodes \(n\) and the communication bandwidth \(b\) is fixed beforehand, and we want to compute a function \(f: \{0, 1\}^{nL} \to \{0, 1\}\), where \(L\) is an integer; each node receives \(L\) private input bits, and we want all nodes to output the same result \(y \in \{0, 1\}\). We define an \((n, b, L, t)\)-protocol \(P\) to be an algorithm that works in this setting and computes an output \(P(x_1, x_2, \ldots, x_n) \in \{0, 1\}\) in \(t\) rounds. Since the parameters are fixed, there are only a finite number of different \((n, b, L, T)\)-protocols, and this number can be bounded by standard counting arguments:

**Lemma 1** ([1]). The number of different \((n, b, L, t)\)-protocols is at most

\[2^{2bm^22^{L+4d(n-1)}}.\]

By contrast, the number of functions \(f: \{0, 1\}^{nL} \to \{0, 1\}\) is \(2^{2^{nL}}\), so this implies that for sufficiently large \(n\), most such functions do not have a \((n, b, L, t)\)-protocol when \(t < L/b - 1\) [1].

### 4 Time hierarchy

We start by proving the deterministic time hierarchy theorem: there are problems of essentially all complexities in the congested clique model.

**Theorem 2.** Let \(f, g: \mathbb{N} \to \mathbb{N}\) be computable functions such that \(S(n) = o(T(n))\) and \(T(n) = O(n/\log n)\). Then

\[\text{CLIQUE}(O(S(n))) \subsetneq \text{CLIQUE}(O(T(n))).\]

**Proof.** We prove the theorem by constructing a language

\[L \in \text{CLIQUE}(O(T(n))) \setminus \text{CLIQUE}(O(S(n))).\]

For convenience, let us assume that \(T(n) < n/(4 \log n)\) for all sufficiently large \(n\). Now, for all sufficiently large \(n\), we define the set of \(n\)-node graphs that belong to \(L\) as follows:

- Fix \(L = T(n)\log n \leq \lfloor n/2 \rfloor\), and fix a function \(f_n: \{0, 1\}^{nL} \to \{0, 1\}\) that does not have a \((n, \log n, L, T(n)/2)\)-protocol; by Lemma 1, such a function exists. Moreover, we can select \(f_n\) to be a first function that satisfies this condition under the lexicographical ordering when interpreting functions \(\{0, 1\}^{nL} \to \{0, 1\}\) as bit vectors of length \(2^{nL}\).
Let $G$ be a graph on $n$ nodes and let $x_v$ be the $L$-bit prefix of the input bit vector that node $v$ receives when $G$ is the input graph. We set $G \in L$ if $f_n(x_1, x_2, \ldots, x_n) = 1$, and $G \notin L$ otherwise.

First, we observe that $L \in \text{CLIQUE}(O(T(n)))$. That is, we can decide if the input graph $G$ belongs to $L$ in time $T(n)$ as follows:

1. Each node $v$ broadcasts the first $L = T(n)\log n$ bits of its input – that is, the vector $x_v$ – to all other nodes. This takes $T(n)$ rounds.

2. Each node $v$ uses local computation to find the function $f_n$ as specified above; this can be done by exhaustively enumerating all functions $f : \{0, 1\}^{nL} \rightarrow \{0, 1\}$ and all $(n, \log n, L, T(n))$-protocols. Each node then locally computes the value $f_n(x_1, x_2, \ldots, x_n)$ and outputs it.

It remains to show that $L \notin \text{CLIQUE}(O(S(n)))$. Assume for contradiction that there is an algorithm that solves $L$ in time $O(S(n))$. This implies that for any sufficiently large $n$, there is an $(n, \log n, L, O(S(n)))$-protocol $P_n$ for $f_n$. However, we have that $S(n) = o(T(n))$, so by the choice of $f_n$ the protocol $P_n$ cannot exist.

\section{Nondeterminism}

\textbf{Nondeterministic complexity classes.} A labelling $z$ of size $k$ is a mapping that assigns each node $v \in V$ a label $z_v \in \{0, 1\}^*$ of length at most $k$. A \textit{nondeterministic congested clique algorithm} $A$ is an algorithm that takes as an input, in addition to the input graph $G$, a labelling $z$ of size $S(n)$ for some computable function $S : \mathbb{N} \rightarrow \mathbb{N}$; we say that $S(n)$ is the labelling size of $A$. We can think of $z$ as the sequence of nondeterministic choices made by $A$, or alternatively as a certificate provided by an external prover. We say that $A$ decides the language $L$ if for all graphs $G$,

$$G \in L \quad \text{if and only if} \quad \exists z : A(G, z) = 1,$$

where $z$ is a labelling of size $S(n)$.

For a computable function $T : \mathbb{N} \rightarrow \mathbb{N}$, we define the complexity class $\text{NCLIQUE}(T(n))$ as the set of languages $L$ such that there exists a nondeterministic algorithm $A$ with running time of $T(n)$ rounds that decides $L$. For a family of functions $C$, we again define

$$\text{NCLIQUE}(C) = \bigcup_{T \in C} \text{NCLIQUE}(T(n)),$$

where $T$ ranges over computable functions in $C$.

\textbf{NCLIQUE normal form.} While the definition of $\text{NCLIQUE}(T(n))$ allows the algorithms to use an essentially arbitrary amount of nondeterministic bits, we show that nondeterministic bits are only useful, roughly speaking, as long as they can be communicated to other nodes by the algorithm. More precisely, we prove that any nondeterministic algorithm can be converted to a normal form:

\textbf{Theorem 3.} If $L \in \text{NCLIQUE}(T(n))$, then there is a nondeterministic algorithm $B$ that decides $L$ with running time $T(n)$ and labelling size $O(T(n)n \log n)$.

\textbf{Proof.} Let $A$ be the algorithm certifying that $L \in \text{NCLIQUE}(T(n))$. We say that a communication transcript of an execution of $A$ of node $v$ is a bit vector consisting of all messages sent and received by $v$ during the execution of $A$. Clearly, a communication transcript of a node $v$ has length $O(T(n)n \log n)$.

We now define an algorithm $B$ that works as follows on input $(G, z)$:
(1) Each node \( v \in V \) checks that their label \( z_v \) is a valid communication transcript of length \( O(T(n)n \log n) \) (if not, reject).

(2) Nodes verify that their labels are consistent with each other; this can be done in \( T(n) \) rounds by simply replaying the transcripts and checking that all the received messages agree with the transcript (if not, reject).

(3) Each node \( v \in V \) locally tries all possible local labels of size at most \( S(n) \), where \( S(n) \) is the labelling size of \( A \), to see if there is a label \( z'_v \) so that the execution of \( A \) with local label \( z'_v \) and the local input of node \( v \) agrees with the transcript \( z_v \) and accepts (if not, reject; otherwise accept).

Clearly \( B \) runs in \( T(n) \) rounds. If there is a labelling \( z' \) such that \( A(G, z') = 1 \), then using the transcripts from this execution of \( A \) as the labelling \( z \) clearly gives \( B(G, z) = 1 \). On the other hand, if there is a \( z \) such that \( B(G, z) = 1 \), then there are local labels \( z'_v \) for each \( v \in V \) such that \( A(G, z') = 1 \).

\[ \square \]

**Nondeterministic time hierarchy.** As an application of the normal form theorem, we can extend the time hierarchy theorem to the nondeterministic congested clique. In fact, we prove a somewhat stronger statement:

**Theorem 4.** Let \( f, g : \mathbb{N} \to \mathbb{N} \) be computable functions such that \( S(n) = o(T(n)) \) and \( T(n) = O(n / \log n) \). Then there is a decision problem \( L \) such that

\[ L \notin \text{NCLIQUE}(O(S(n))) \quad \text{and} \quad L \in \text{CLIQUE}(O(T(n))) . \]

**Proof.** We say that a \((n,b,M+L,t)-protocol\) is a nondeterministic protocol for function \( f : \{0,1\}^{nL} \to \{0,1\} \) if for all \( x \in \{0,1\}^{nL} \) it holds that \( f(x_1,x_2,\ldots,x_n) = 1 \) if and only if there is \( z \in \{0,1\}^{nM} \) such that \( P(z_1x_1,z_2x_2,\ldots,z_nx_n) = 1 \).

Now we construct a decision problem \( L \in \text{CLIQUE}(O(T(n))) \setminus \text{NCLIQUE}(O(S(n))) \) using the same construction as in the proof of Theorem 2, with minor modifications as follows. Let \( L = T(n) \log n \), and let \( M = \frac{1}{4}T(n)n \log n \). We select the functions \( f_n : \{0,1\}^{nL} \to \{0,1\} \) in the construction of \( L \) with the extra constraint that \( f_n \) does not have a nondeterministic \((n, \log n, M+L, T(n)/4))-protocol; this is possible for sufficiently large \( n \), since then

\[ 2^{M+L+T(n)(n-1)} \log n \leq \left( 1 + \frac{1}{n} \right) T(n)n \log n \leq \frac{3}{4}T(n)n \log n = \frac{3}{4}nL, \]

and thus by Lemma 1 the number of \((n, \log n, M+L, T(n)/4))-protocols is \( 2^{o(2^{nL})} \).

The problem \( L \) constructed using the functions \( f_n \) is clearly in \( \text{CLIQUE}(T(n)) \) using the same argument as in the proof of Theorem 2. On the other hand, if \( L \in \text{NCLIQUE}(O(S(n))) \), then by Theorem 3 there is a nondeterministic algorithm for \( L \) with running time \( O(S(n)) \) and labelling size \( O(S(n)n \log n) \). But this implies that the functions \( f_n \) have nondeterministic \((n, \log n, L + O(S(n)n \log n), O(S(n)))\)-protocols, which is not possible for large \( n \) by the choice of \( f_n \), since we have \( S(n) = o(T(n)) \).

\[ \square \]

Since \( \text{CLIQUE}(O(T(n))) \subseteq \text{NCLIQUE}(O(T(n))) \), a time hierarchy theorem for the nondeterministic congested clique follows immediately from Theorem 4.

**Corollary 5.** Let \( f, g : \mathbb{N} \to \mathbb{N} \) be computable functions such that \( S(n) = o(T(n)) \) and \( T(n) = O(n / \log n) \). Then

\[ \text{NCLIQUE}(O(S(n))) \subseteq \text{NCLIQUE}(O(T(n))) . \]
6 Constant-round decision

Constant-round nondeterministic clique. The class NCLIQUE(\(O(1)\)) is a natural analogue of class NP in the congested clique; it contains decision versions of most natural problems considered in the congested clique setting. It is also easy to see that NCLIQUE(\(O(1)\)) contains many NP-complete decision problems, such as \(k\)-colouring and Hamiltonian paths. By Theorem 4, we also know that there are problems that can be solved in slightly superconstant time in the congested clique, but are not in NCLIQUE(\(O(1)\)). However, our lower bound techniques are not sufficient to show that

\[
\text{CLIQUE}(O(1)) \neq \text{NCLIQUE}(O(1)) ;
\]

in a sense, this is the congested clique analogue of the P vs. NP question.

However, we can interpret Theorem 3 to state that there is a canonical problem family for NCLIQUE(\(O(1)\)). Specifically, we say that a neighbourhood constraint \(\mathcal{C}\) is a computable mapping that for any number of nodes \(n\), any edge \(\{u, v\}\), and any edge-neighbourhood \(\partial(u)\) of \(u\), gives a set \(\mathcal{C}_{n,u,v,\partial(u)}\) of allowed \(O(\log n)\)-bit labels for the edge \(\{u, v\}\). An edge labelling problem is defined by an edge neighbourhood constraint \(\mathcal{C}\); given a graph \(G\), find a labelling \(\ell\) of all edges of the clique with labels \(\ell(u, v)\) of size \(O(\log n)\) so that the labels satisfy the local constraints at all nodes, that is

\[
\ell(u, v) \in \mathcal{C}_{n,u,v,\partial(u)} \quad \text{and} \quad \ell(u, v) \in \mathcal{C}_{n,v,u,\partial(v)}
\]

for all \(u\) and \(v\).

By Theorem 3, any problem with NCLIQUE(\(O(1)\)) algorithm \(A\) can be interpreted as an edge labelling problem: the edge labels are defined as the set of valid communication transcripts of an accepting run of \(A\). This gives us a limited notion of completeness for NCLIQUE(\(O(1)\)):

**Theorem 6.** We have NCLIQUE(\(O(1)\)) \(\subseteq\) CLIQUE(\(O(T(n))\)) if and only if all edge labelling problems can be solved deterministically in \(O(T(n))\) rounds.

In particular, we have CLIQUE(\(O(1)\)) = NCLIQUE(\(O(1)\)) if and only if all edge labelling problems can be solved deterministically in \(O(1)\) rounds. However, it seems that identifying a single graph decision problem that is “complete” for NCLIQUE(\(O(1)\)) is difficult; in essence, we would have to work reductions running in constant time, which makes the use of any sort of gadget constructions extremely difficult.

Constant-round decision hierarchy. Whereas NCLIQUE(\(O(1)\)) is the congested clique analogue of NP, we can extend this analogue to the polynomial hierarchy by adding more quantifiers, that is, by allowing the nodes to alternate between nondeterministic and co-nondeterministic choices; similar ideas has been studied in the context of local verification [2, 14, 42].

Formally, we say that a \(k\)-labelling algorithm \(A\) of labelling size \(S(n)\) is a constant-round congested clique algorithm that takes as an input \(k\) labellings \(z_1, z_2, \ldots, z_k\) of size at most \(S(n)\). We define the class \(\Sigma_k\) as the set of languages \(L\) for which there exists a \(k\)-labelling algorithm \(A\) of labelling size \(S(n)\) such that

\[
G \in L \quad \text{if and only if} \quad \exists z_1 \forall z_2 \ldots Q z_k : A(G, z_1, z_2, \ldots, z_k) = 1,
\]

where \(z_1, z_2, \ldots, z_k\) are labellings of size at most \(S(n)\) and \(Q\) is the universal quantifier if \(k\) is even and the existential quantifier if \(k\) is odd. Similarly, we define \(\Pi_k\) as the set of languages \(L\) for which there exists a \(k\)-labelling algorithm \(A\) of labelling size \(S(n)\) such that

\[
G \in L \quad \text{if and only if} \quad \forall z_1 \exists z_2 \ldots Q z_k : A(G, z_1, z_2, \ldots, z_k) = 1,
\]
where $z_1, z_2, \ldots, z_k$ are labellings of size at most $S(n)$ and $Q$ is the existential quantifier if $k$ is even and the universal quantifier if $k$ is odd. Finally, we define $\Delta_k = \Sigma_k \cup \Pi_k$.

At the first level of the hierarchy we have the class $\Sigma_1 = \text{NCLIQUE}(O(1))$; by Theorem 3 we know that restricting the labelling size to $O(n \log n)$ gives us the same class as unlimited labelling size. A natural question is then if the same phenomenon happens at the higher levels of the hierarchy?

Turns out this is not the case; we will consider two different versions of the constant-round decision hierarchy:

- **Unlimited hierarchy**: the hierarchy $(\Sigma_k, \Pi_k)_{k=1}^\infty$ as defined above, with arbitrary labelling size allowed.
- **Logarithmic hierarchy**: the hierarchy $(\Sigma_k^{\log}, \Pi_k^{\log})_{k=1}^\infty$ defined otherwise as above, but the algorithm $A$ is required to have labelling size of $O(n \log n)$ – in other words, $O(\log n)$ bits per edge.

In a sense, these correspond to the different LOCAL model hierarchies studied by Feuilloley et al. [14] (the logarithmic hierarchy) and Balliu et al. [2] (the unlimited hierarchy).

**Basic properties.** We first note some basic properties of the constant-round hierarchies. Trivially, we have that

$$\Sigma_k \subseteq \Delta_k \subseteq \Sigma_{k+1} \quad \text{and} \quad \Pi_k \subseteq \Delta_k \subseteq \Pi_{k+1},$$

and thus also

$$\Pi_k \subseteq \Sigma_{k+1} \quad \text{and} \quad \Sigma_k \subseteq \Pi_{k+1}.$$ 

Moreover, since the communication graph is fully connected, we can globally complement the output of a decision algorithm; thus, if a decision problem $L$ is in $\Sigma_k$, then the complement language $\bar{L}$ is in $\Pi_k$, and vice versa. These observations also hold for the logarithmic version of the constant-round hierarchy.

**Unlimited hierarchy** $(\Sigma_k, \Pi_k)_{k=1}^\infty$. For the unlimited hierarchy, we obtain an essentially complete characterisation. By Theorem 4, we know there are problems that are not on the first level. Moreover, in a similar manner as happens with the LOCAL hierarchy of Balliu et al. [2], we show that the unlimited hierarchy collapses to the second level:

**Theorem 7.** All decision problems $L$ are in $\Sigma_2 = \Pi_2$.

**Proof.** Let $L$ be a decision problem. To see that $L \in \Pi_2$, consider the following algorithm $A$:

1. The existential quantifier is used to guess a graph $G'_v$ in each node $v$, using $n^2$ bits per node.

2. The universal quantifier is used to verify that all guesses $G'_v$ equal the input graph $G$: each node $v$ uses $O(\log n)$ universal bits to pick a single bit from the encoding of $G'_v$ and broadcasts this bit and its index to all other nodes. All nodes $u$ verify that the information broadcast by other nodes is consistent with their guess of $G'_u$ and their local view of $G$ (if not, reject).

3. Each node $v$ locally checks if $G'_v \in L$ (if this holds for all nodes, accept; otherwise reject).
Theorem 8. There is a decision problem \( L \) such that \( L \notin \bigcup_{k=0}^{\infty} \Sigma_k^{\text{log}} \).

Proof. Again, we use the same general proof technique as in the proofs of Theorems 2 and 4. Fix a computable function \( T(n) = \omega(n) \); let \( L = (T(n))^{2\log_2 n} \) and \( M = \frac{1}{4} T(n) n \log n \). Otherwise using the same construction for the language \( L \) as before, we select the functions \( f_n : \{0,1\}^{nL} \to \{0,1\} \) so that for any \( k \leq T(n) \), there is no \((n, \log n, kM + L, (T(n))^{2/4})\)-protocol that \( \Sigma_k^{\text{log}} \)-computes \( f_n \); this is possible for sufficiently large \( n \), since then

\[
kM + L + \frac{1}{4} (T(n))^2 (n-1) \log n < \frac{3}{4} (T(n))^2 n \log n = \frac{3}{4} nL,
\]

and thus by Lemma 1 the number of \((n, \log n, kM + L, (T(n))^{2/4})\)-protocols is \( 2^{o(2^{nL})} \).

If \( L \in \Sigma_k^{\text{log}} \) for some \( k \), then there is a \( \Sigma_k \) algorithm for \( L \) with running time \( O(1) \) and labelling size \( O(n \log n) \). But this implies that there is an \((n, \log n, O(kn \log n) + L, O(1))\)-protocol that \( \Sigma_k \)-computes \( L \); this is not possible for large \( n \) by the choice of \( f_n \), since we have \( T(n) = \omega(1) \). \( \square \)

The proof of Theorem 8 actually implies that there are problems with only slightly super-constant deterministic complexity that are not on any level of the hierarchy. However, our lower bound technique does not appear to be sufficiently refined to separate different levels of the logarithmic constant-round hierarchy, so it remains open whether the hierarchy has infinitely many levels.

7 Fine-grained complexity

Problem exponent. In the following, we will consider concrete problems that are not necessarily decision problems, and allow more generally problems defined in terms of weighted graphs and matrices. For a problem \( L \), we define the exponent of \( L \) as

\[
\delta(L) = \inf \{ \delta \in [0,1] : L \text{ can be solved in } O(n^\delta) \text{ rounds} \}.
\]

The basic idea is that the problem exponent captures the polynomial complexity of the problem, and we can compare the relative complexity of problems by comparing their exponents.

Relationships between problems. In Figure 1, we map out some known relationships between prominent problems in the congested clique using this framework:

- Relationships between \( k \)-independent set detection (\( k \)-IS), maximum independent set (\( \text{MaxIS} \)), maximum clique (\( \text{MaxCLIQUE} \)) and minimum vertex cover (\( \text{MinVC} \)) are trivial. The upper bound for \( k \)-independent set is due to Dolev et al. [11].
Figure 1: Relationships between selected problems in the congested clique model.

- Relationships between 3-independent set detection and Boolean matrix multiplication (Boolean-MM) as well as between approximate weighted all-pairs shortest paths problem and ring matrix multiplication (Ring-MM) follow from the work of Censor-Hillel et al. [6]; they also give the upper bound $\delta$(Ring-MM) $\leq 1 - 2/\omega$, where $\omega < 2.3728639$ is the matrix multiplication exponent [32].

- The relationship between $k$-colouring ($k$-COL) and maximum independent set follows from an easy reduction [37]: replace each vertex $v$ with $k$ copies $v_1, \ldots, v_k$ connected into a clique, and connect $v_i$ and $u_i$ if the edge $\{v, u\}$ is present in the original graph. Clearly, the new graph has independent set of size $n$ if and only if the original graph is $k$-colourable. For constant $k$, the blowup from implementing this reduction is constant.

- The relationship between $(2 - \varepsilon)$-approximate weighted undirected all-pairs shortest paths problem ($(2 - \varepsilon)$-APX-APSP) and Boolean matrix multiplication follows from a reduction by Dor et al. [12].

We note that one challenge involved in this approach is that the congested clique setting requires the use of extremely fine-grained reductions; we are essentially allowed only $n^{o(1)}$ factor blowups in the reductions. By contrast, most known reductions between NP-complete problems have polynomial blowup, making them useless in this setting.

One potentially fruitful perspective is to consider for which pairs of problems we cannot prove reductions. For example, the reduction from Boolean matrix multiplication to $(2 - \varepsilon)$-approximate APSP breaks down if we consider 2-approximate APSP instead. Indeed, we know that constant-approximation APSP can be solved faster than the current matrix multiplication upper bound, using the spanner constructions of Censor-Hillel et al. [7], so conjecturing a faster-than-matrix-multiplication upper bound for 2-approximate APSP does not seem unreasonable.

8 Conclusions

**CLIQUE**(O(1)) vs. NCLIQUE(O(1)). As a major open question, we highlight the lack of separation between constant-round deterministic and nondeterministic congested clique. Given that NCLIQUE(O(1)) contains, among others, NP-complete problems, it seems reasonable to conjecture that

$$\text{CLIQUE}(O(1)) \neq \text{NCLIQUE}(O(1)),$$

but it is not clear how we should approach proving such separation. Indeed, it would be interesting even if we could prove this conditional on a centralised complexity assumption, such as $P \neq NP$. 

11
Randomness. In this work, we have focused on deterministic and nondeterministic computation; however, there are problems in the congested clique model where the best randomized upper bounds are significantly better than the best deterministic upper bounds, e.g. minimum spanning tree [19, 36]. Thus, a possible extension of the present work is to study the randomised complexity landscape of the congested clique. Indeed, the counting arguments of Applebaum et al. [1] extend to randomised protocols. Likewise, Theorem 4 implies that there are problems that cannot be solved in $O(S(n))$ rounds with one-sided Monte Carlo algorithms, but can be solved in $O(T(n))$ rounds deterministically for $S(n) = o(T(n))$, as the Monte Carlo algorithm can be converted to a nondeterministic algorithm.

Acknowledgements. This work was supported in part by the Academy of Finland, Grant 285721. We thank Parinya Chalermsook, Juho Hirvonen, Petteri Kaski and Christopher Purcell for comments and discussions.

References


