Fast distributed approximation algorithms for vertex cover and set cover in anonymous networks

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Braunschweig,
29 November 2010

Joint work with Matti Åstrand
Vertex cover problem

- **Vertex cover $C$** for a graph $G$:
  - Subset $C$ of nodes that “covers” all edges of the graph
  - Each edge has at least one endpoint in $C$
- Can we find a *small* vertex cover?
Vertex cover problem

- Classical NP-hard optimisation problem
  - Simple 2-approximation algorithm: endpoints of a \textit{maximal matching}
  - No polynomial-time algorithm with approximation factor 1.999 known
Research question

- **Distributed** approximation algorithms for vertex cover
  - Find a small vertex cover in any communication network
  - Best possible approximation ratio
  - As fast as possible: running time independent of $n$
  - Weakest possible models: no randomness, no unique node identifiers

- Let’s first define the models...
Distributed algorithms

- Communication graph $G$
- Node = computer
- Edge = communication link
- Computers exchange messages and finally decide whether they are in vertex cover $C$
  - “Local output”, 0 or 1
Distributed algorithms

- All nodes are identical, run the same algorithm
- We can choose the algorithm
- An adversary chooses the structure of $G$
- Our algorithm must produce a valid vertex cover in any graph $G$
Model 1: Unique identifiers

- The “standard model”
- Node identifiers are a subset of $1, 2, \ldots, \text{poly}(n)$
- Subset chosen by adversary
Model 2: Port-numbering model

- No unique identifiers
- A node of degree $d$ can refer to its neighbours by integers 1, 2, ..., $d$
- Port-numbering chosen by adversary
Model 3: Broadcast model

- No identifiers, no port numbers
- A node has to send the same message to each neighbour
- A node does not know which message was received from which neighbour (multiset)
Deterministic distributed algorithms for vertex cover

• Guaranteed approximation ratios?
  • E.g., 2-approximation of minimum vertex cover = at most 2 times as large as the smallest vertex cover

• Fast?
  • Time = number of communication rounds
  • $n$ = number of nodes
  • $\Delta$ = maximum degree

• In weak models of distributed computing?
Deterministic distributed algorithms for vertex cover: approximation ratios

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Deterministic distributed algorithms for vertex cover: approximation ratios

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Broadcast model
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Maximal matching
(Panconesi & Rizzi 2001)
Deterministic distributed algorithms for vertex cover: approximation ratios

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Near-maximal edge packing (Khuller et al. 1994)

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**Deterministic LP rounding** (Kuhn et al. 2006)
## Deterministic distributed algorithms for vertex cover: approximation ratios

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- Czygrinow et al. 2008
- Lenzen & Wattenhofer 2008
Deterministic distributed algorithms for vertex cover: approximation ratios

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Trivial (cycles)
Deterministic distributed algorithms for vertex cover: approximation ratios

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- Port numbering
- Unique identifiers
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**DISC 2009**
Deterministic distributed algorithms for vertex cover: approximation ratios

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Latest results + faster and more general solution here

- Broadcast model
- Port numbering
- Unique identifiers
Deterministic distributed algorithms for vertex cover: approximation ratios

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Let’s study this case first...
Vertex cover in the port-numbering model

- Convenient to study a more general problem: minimum-weight vertex cover
  - More general problems are sometimes easier to solve?

Notation: \( w(v) = \text{weight of } v \)
Edge packings and vertex covers

- **Edge packing**: weight $y(e) \geq 0$ for each edge $e$
  - Packing constraint: $y[v] \leq w(v)$ for each node $v$, where $y[v] =$ total weight of edges incident to $v$

edge packing $\approx$ fractional matching (LP relaxation)
Edge packings and vertex covers

- Node $v$ is **saturated** if $y[v] = w(v)$
  - Total weight of edges incident to $v$ is **equal** to $w(v)$, i.e., the packing constraint holds with equality.

![Graph with edge weights and vertex labels]

- $y[v] = w(v)$
- $y[v] < w(v)$
Edge packings and vertex covers

• Edge $e$ is **saturated** if at least one endpoint of $e$ is saturated
  • Equivalently: edge weight $y(e)$ can’t be increased

2 + $\varepsilon$ would violate a packing constraint
Edge packings and vertex covers

- **Maximal edge packing**: all edges saturated
  \[\Leftrightarrow\] none of the edge weights \(y(e)\) can be increased
  \[\Leftrightarrow\] saturated nodes form a vertex cover
Edge packings and vertex covers

- **Maximal edge packing**: all edges saturated
  \[ \iff \text{saturated nodes form a vertex cover} \]
  - … and saturated nodes are 2-approximation of minimum-weight vertex cover (Bar-Yehuda & Even 1981)

- How to find a maximal edge packing…?
  - Phase II: if phase I fails to saturate an edge \( e = \{u, v\} \), we can *break symmetry* between \( u \) and \( v \); exploit it!
Finding a maximal edge packing: phase I

- \( y[v] \) = total weight of edges incident to node \( v \)
- **Residual capacity** of node \( v \): \( r(v) = w(v) - y[v] \)
- Saturated node: \( r(v) = 0 \)

![Graph diagram with node and edge weights and residual capacities]
Finding a maximal edge packing: phase I

Start with a trivial edge packing $y(e) = 0$
Finding a maximal edge packing: phase I

Each node $v$ offers $r(v)/\text{deg}(v)$ units to each incident edge

- $r(v): 9$
- $w(v): 9$
- offer: 9
Finding a maximal edge packing: phase I

Each edge **accepts** the smallest of the 2 offers it received

Increase $y(e)$ by this amount

- Safe, can’t violate packing constraints
Finding a maximal edge packing: phase I

Update residuals...
Finding a maximal edge packing: phase I

Update residuals, discard saturated nodes and edges...
Finding a maximal edge packing: phase I

Update residuals, discard saturated nodes and edges, repeat...

Offers...
Finding a maximal edge packing: phase I

Update residuals, discard saturated nodes and edges, repeat...

Offers...

Increase weights...
Finding a maximal edge packing: phase I

Update residuals, discard saturated nodes and edges, repeat...

Offers...

Increase weights...

Update residuals...
Finding a maximal edge packing: phase I

Update residuals, discard saturated nodes and edges, repeat...

Offers...
Increase weights...
Update residuals and graph, etc.
Finding a maximal edge packing: phase I

We are making some progress towards finding a maximal edge packing...

But this is **too slow**!

How to make it faster?
Finding a maximal edge packing: colouring trick

- Offer is a local minimum:
  - Node will be saturated
  - And all edges incident to it will be saturated as well

Residual capacity was 8, will be 0
Finding a maximal edge packing: colouring trick

- Offer is a local minimum:
  - Node will be **saturated**
- Otherwise there is a neighbour with a different offer:
  - Interpret the offer sequences as colours
  - Nodes \( u \) and \( v \) have different colours: \( \{u, v\} \) is **multicoloured**
Finding a maximal edge packing: colouring trick

- Progress guaranteed:
  - On each iteration, for each node, at least one incident edge becomes saturated or multicoloured
  - Such edges are be discarded in phase I; maximum degree $\Delta$ decreases by at least one
  - Hence in $\Delta$ rounds all edges are saturated or multicoloured
Finding a maximal edge packing: colouring trick

- Colours are sequences of $\Delta$ offers (rational numbers)
  - Assume that node weights are integers $1, 2, \ldots, W$
  - Then offers are rationals of the form $q/(\Delta!)^\Delta$ with $q \in \{1, 2, \ldots, W(\Delta!)^\Delta\}$

(2, 2/3, 1/6, 1/12)

(2, 2/3, 1/6, 1/24)
Finding a maximal edge packing: colouring trick

- Colours are sequences of $\Delta$ offers (rational numbers)
  - Assume that node weights are integers 1, 2, ..., $W$
  - Then offers are rationals of the form $q/(\Delta!)^\Delta$ with $q \in \{1, 2, ..., W(\Delta!)^\Delta\}$
  - $k = (W(\Delta!)^\Delta)^\Delta$ possible colours, replace with integers 1, 2, ..., $k$
Finding a maximal edge packing: phase II

- Proper $k$-colouring of the unsaturated subgraph
- Orient from lower to higher colour
- Partition in $\Delta$ forests
  - Use Cole-Vishkin (1986) style colour reduction algorithm
- Use colour classes to saturate edges
- $O(\Delta + \log^* W)$ rounds
Finding a maximal edge packing: summary

- Maximal edge packing and 2-approximation of vertex cover in time $O(\Delta + \log^* W)$
  - $W = \text{maximum node weight}$

- Unweighted graphs: running time simply $O(\Delta)$, independent of $n$

- Everything can be implemented in the port-numbering model
Vertex cover and set cover in anonymous networks: summary

• 2-approximation of vertex cover in time $O(\Delta)$ in the port-numbering model
  • Idea: consider a more general problem, minimum-\textit{weight} vertex cover

• 2-approximation of vertex cover in time $\text{poly}(\Delta)$ in the broadcast model?
  • Idea: consider a more general problem, minimum-weight \textit{set cover}!
Take-home messages

• Algorithms that we saw today are strictly local
  • Running time independent of the number of nodes
  • Output of a node depends only on its local neighbourhood
  • Very efficient, can be used in arbitrarily large networks
  • Deterministic, highly fault-tolerant

• There are non-trivial graph problems that can be solved with strictly local algorithms!
  • More: www.cs.helsinki.fi/jukka.suomela/local-survey