Labelling grids locally

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Setting the scene

- Computer scientists study what can be computed with limited resources, e.g. space and time.
- One interpretation is that we are studying just how complex time and space can be.
- Philosophically, the conjecture $P \subset \text{PSPACE}$ proposes that space is more complex than time.
- Other examples include recent work on the black hole firewall paradox.
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Aims

- Our broad goal is to take a similar look at reality through the lens of distributed computing.
- In distributed computing, our limited resources are information about the input and communication.
- We want to understand the distributed computational capabilities of reality.
Grids

- Grids are a reasonable place to start and are nice to reason about.
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Our model

- A collection of nodes communicate in synchronous rounds and try to solve a problem.
- Each node sits at a vertex of a graph $G$ and in each round communicates with its neighbours.
- The nodes may have some input (e.g., a unique identifier) but do not know $G$ initially.
- Local computation is free; the complexity measure is the number of rounds.
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- Input graph is an $n \times n$ grid with a consistent orientation unless otherwise specified.
- Nodes have unique identifiers and must output a label according to some set of rules.
- We restrict ourselves to labellings that are *locally checkable*: validity in the constant radius around each node implies global validity; e.g., vertex colouring, edge colouring, maximal independent set (MIS)
Warm up: one dimension

Theorem

On an oriented cycle, every locally checkable labelling problem that cannot be solved in $O(\log^* n)$ time requires $\Omega(n)$ time.

Idea: for suitable constant $k$, we find a MIS in $G_k$ in $O(\log^* n)$ time and try to fix a particular labelling for each possible gap.
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Figure: An MIS in $G^3$
Warm up: one dimension

**Theorem**
*On an oriented cycle, every solvable locally checkable labelling problem has asymptotic complexity* $O(1)$, $\Theta(\log^* n)$ or $\Theta(n)$ *time.*

**Definition**
A vertex $v$ in a directed graph is *flexible* if there exists a constant $k$ such that for all $k' > k$ there is a circuit of length $k$ that begins and ends at $v$.

*Figure: Locally optimal cut*
Warm up: one dimension

**Theorem**

*On an oriented cycle, every solvable locally checkable labelling problem has asymptotic complexity $O(1)$, $\Theta(\log^* n)$ or $\Theta(n)$ time.*

**Proof (Sketch).**

If the *neighbourhood graph* of $\Pi$ has a flexible label we can use the MIS. If there is no such label, then for some $b$ nodes at distance $0 \bmod b$ must have the same label.

![Locally optimal cut](image-url)  
*Figure: Locally optimal cut*
Speed up

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Idea: If we have an algorithm $A$ that runs in $o(n)$ time, nodes running $A$ don’t see the edges of the grid. Maybe we can simulate $A$ in a bigger grid. The problem is the unique identifiers.
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Given algorithm $A$ that runs in $o(r)$ on $r \times r$ grids we want an algorithm that runs in $O(\log^* n)$ for $n \times n$ grid $G$. 

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Proof (Sketch).

Given algorithm \( A \) that runs in \( o(r) \) on \( r \times r \) grids we want an algorithm that runs in \( O(\log^* n) \) for \( n \times n \) grid \( G \).

- Pick suitable constants \( r, k \).
- Find an MIS in \( G^k \), and use it to pick locally unique identifiers for \( r \times r \) neighbourhoods.
- Simulate \( A \).
Undecidability

Question
Can we characterise the classes of LCLs having asymptotic complexity $O(1), \Theta(\log^* n), \Theta(n)$?
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Not in general - whether a given LCL has complexity $\Theta(\log^* n)$ or $\Theta(n)$ on grids is undecidable.
If we know that a problem has $\Theta(\log^* n)$ complexity on grids (or we make a lucky guess) we can find such an algorithm.

- For each $r, k$ we enumerate all radius $r$ neighbourhoods that represent possible fragments of an MIS in $G^k$.
- Construct the neighbourhood graph - an algorithm is a labelling of the neighbourhood graph.
- Use SAT solvers to find a labelling.
A 2-colouring may not exist in a grid, and is inherently a global problem. A 5-colouring of a grid is a $\Delta + 1$-colouring

- 2-colouring: $\Theta(n)$
- 3-colouring: ??
- 4-colouring: ??
- 5-colouring: $\Theta(\log^* n)$
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Results: two labels

\textit{a, b-labelling}: black nodes have at least \textit{a} white neighbours, white nodes have at least \textit{b} black neighbours.
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\textbf{Figure:} Asymptotic complexity of \(a, b\)-labellings for oriented cycles

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Summary

- For directed grids all LCL problems have a $O(\log^* n)$ upper bound or $\Omega(n)$ lower bound.
- 4-colouring is $\Theta(\log^* n)$, 3-colouring is $\Theta(n)$.
- $O(\log * n)$ algorithms can be synthesised.
Questions

- Interpretation: what’s the moral of the story?
- Connections to physics and real world systems.
- What next?