DDA 2010, lecture 5:
Weak colouring and other tricks

- Symmetry *can* be broken very fast if nodes have odd degrees...
  - ... but we need *port numbering and orientation*
DDA 2010, lecture 5a: Port numbering and orientation

- A new model
  - stronger than the port-numbering model
  - weaker than networks with unique identifiers
Introduction

• How could we design algorithms that are faster than Cole-Vishkin? Constant-time algorithms?
  • if we try to exploit the numerical values of unique identifiers, we will usually get running times $\Omega(\log^* n)$ or worse
  • what if we just used the relative order of unique identifiers?
  • let’s have a look at a model in which each pair of neighbours is ordered, and see what kinds of problems can be solved…
Port-numbering and orientation

- A node of degree \( d \) can refer to its neighbours by integers 1, 2, ..., \( d \)
- Each edge has an orientation
  - ends labelled: head, tail
- Port-numbering and orientation chosen by adversary
Port-numbering and orientation

- If you have unique identifiers or colouring, you can easily find an orientation
  - orient from smaller to larger ID (or colour)
  - we used this trick in lecture 2 to construct directed forests
Port-numbering and orientation

- Is this model stronger than port numbering?
- Is this model weaker than unique identifiers?
Port-numbering and orientation

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  - Yes: colouring of 2-node paths is possible

- Is this model weaker than unique identifiers?
Port-numbering and orientation

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- Is this model weaker than unique identifiers?
  - Yes: colouring of 3-cycles is impossible
Port-numbering and orientation

- Can we solve anything non-trivial in this model?
- Looks bad: we can still have symmetric inputs
Port-numbering and orientation

- Can we solve anything non-trivial in this model?
- Looks bad: we can still have symmetric inputs
  - but in all these constructions $\text{indegree} = \text{outdegree}$, and therefore nodes must have even degrees!
DDA 2010, lecture 5b: Weak colouring

- Naor-Stockmeyer (1995):
  - fast symmetry breaking in graphs with indegree ≠ outdegree
Symmetry breaking in graphs with port numbering and orientation

- The simplest case: 1-regular graphs
- Consists of isolated edges, certainly we can break symmetry for each pair of nodes
  - one is “head”, the other one is “tail”
    (head has indegree 1, tail has outdegree 1)
Symmetry breaking in graphs with port numbering and orientation

- In general, we can always label nodes by their (outdegree, indegree) pairs
  - different outdegrees or different indegrees: different labels, symmetry broken
  - only $O(\Delta^2)$ possible labels; easy to reduce using C-V tricks
Symmetry breaking in graphs with port numbering and orientation

• In general, we can always label nodes by their (outdegree, indegree) pairs
• But what if a node and all of its neighbours have identical (outdegree, indegree) pairs?
Symmetry breaking in graphs with port numbering and orientation

• In general, we can always label nodes by their (outdegree, indegree) pairs

• But what if a node and all of its neighbours have identical (outdegree, indegree) pairs?
  
  • we already know that if outdegree = indegree for all nodes, we are in trouble
  
  • but what if we know that outdegree ≠ indegree?
  
  • for example, what if all nodes have degree = 3 and therefore necessarily outdegree ≠ indegree?
Symmetry breaking in graphs with port numbering and orientation

- Simplest case: indegree = 1, outdegree = 2
- Label = outgoing port number in predecessor

![Diagram showing symmetry breaking in a graph with port numbering and orientation](image)
Symmetry breaking in graphs with port numbering and orientation

- Simplest case: indegree = 1, outdegree = 2
- Label = outgoing port number in predecessor
Symmetry breaking in graphs with port numbering and orientation

- Simplest case: indegree = 1, outdegree = 2
- Label = outgoing port number

Can’t have $X = 2$ and $X = 3$
Symmetry broken!
Symmetry breaking in graphs with port numbering and orientation

- We can construct a weak colouring:
  - for each non-isolated node at least one neighbour has different colour
- C-V can be used to reduce the number of colours
Symmetry breaking in graphs with port numbering and orientation

- Indegree = 1, outdegree = 2: *weak colouring*
  - node takes its label from the port numbers of its parent
- Generalisation to any indegree ≠ outdegree?
  - enough to study the case indegree < outdegree
  - then we can reverse the directions and get the same result for indegree > outdegree!
  - let’s present the algorithm in the general case and prove that it finds a weak colouring...
Symmetry breaking in graphs with port numbering and orientation

- General case: indegree < outdegree
- Label = list of outgoing port numbers in all predecessors

![Diagram of graph with port numbering and orientation]
Symmetry breaking in graphs with port numbering and orientation

- General case: indegree < outdegree
- Label = list of outgoing port numbers in all predecessors
Symmetry breaking in graphs with port numbering and orientation

• General case: indegree < outdegree

• Label = list of outgoing port numbers in all predecessors
Symmetry breaking in graphs with port numbering and orientation

• **Lemma**: for each $v$, the successors of $v$ have at least 2 different labels
  
  • Proof: pigeonhole again…
Symmetry breaking in graphs with port numbering and orientation

- E.g., outdegree = 3, indegree = 2:
  - a 2-element list can’t contain all 3 outgoing port numbers of v
  - must have at least 2 different 2-element lists!

These pairs must contain all 3 outgoing port numbers of v
Symmetry breaking in graphs with port numbering and orientation

- General case, outdegree = $s$, indegree = $t$:
  - an $s$-element list can’t contain all $t$ outgoing port numbers of $v$ if $s < t$
  - must have at least 2 different $s$-element lists!

These pairs must contain all 3 outgoing port numbers of $v$
Symmetry breaking in graphs with port numbering and orientation

• **Lemma**: for each \( v \), the successors of \( v \) have at least 2 different labels

• **Corollary**: \( v \) has a successor \( u \) such that \( v \) and \( u \) have different labels
  
  • i.e., we have a weak colouring
  
  • again, we can use C-V to reduce the number of colours
  
  • it is possible to construct a weak 2-colouring; running time is \( O(\log^* \Delta) \), independent of \( n \)
  
  • assumptions: port numbering, indegree \( \neq \) outdegree
Summary

- **Model:** port numbering and orientation

- If outdegree = indegree:
  - we may have a symmetric input
  - in the worst case all nodes will produce the same output

- If outdegree ≠ indegree:
  - symmetry can be broken
  - we can find a weak 2-colouring — very fast!
  - however, we can’t find a (non-weak) colouring