DDA 2010, lecture 1: Introduction

• Synchronous deterministic distributed algorithms

• Two models:
  • Port-numbering model
  • Unique identifiers
Some notational conventions

• Graphs:
  - unless otherwise mentioned, graphs are **undirected** and **simple**
  - graphs are pairs: $G = (V, E)$, $V$ set of nodes, $E$ set of edges
  - undirected edges are unordered pairs: if there is an edge between $u \in V$ and $v \in V$, we have $\{u, v\} \in E$
  - directed edges are ordered pairs, e.g. $(u, v) \in E$
  - $\deg(v) = \text{degree of } v \in V$
Some notational conventions

• Parameters:
  • \( n = |V| \), number of nodes
  • \( \Delta \) is an upper bound on degrees:
    \[
    \text{deg}(v) \leq \Delta \text{ for all } v \in V
    \]

• These are often used in algorithm analysis
  • e.g., “running time \( O(\Delta + \log n) \)”

• Sometimes we assume that \( \Delta \) is a global constant
  • “bounded-degree graphs”, \( \Delta = O(1) \)
DDA 2010, lecture 1a: Port-numbering model

- Synchronous deterministic distributed algorithms in the port-numbering model
- Limited model, we will study extensions later
Distributed algorithms

- Communication graph $G$
- Node = computer
  - e.g., Turing machine, finite state machine
- Edge = communication link
  - computers can exchange messages
Distributed algorithms

- All nodes are identical, run the same algorithm
- We can choose the algorithm
- An adversary chooses the structure of $G$
- Our algorithm must produce a correct output in any graph $G$
Distributed algorithms

• Usually, computational problems are related to the structure of the communication graph $G$
  
  • example: find a maximal independent set for $G$
  
  • the same graph is both the input and the system that tries to solve the problem...
Port-numbering model

- A node of degree $d$ can refer to its neighbours by integers $1, 2, \ldots, d$
- Port-numbering chosen by adversary
Synchronous distributed algorithms

1. Each node reads its own local input
   - Depends on the problem, for example:
     - node weight
     - weights of incident edges
   - May be empty
Synchronous distributed algorithms

1. Each node reads its own local input
2. Repeat synchronous communication rounds
   ...

1
2
2
2
3
1
1
1
2
Synchronous distributed algorithms

1. Each node reads its own local input
2. Repeat synchronous communication rounds until all nodes have announced their local outputs
   • Solution of the problem
Synchronous distributed algorithms

1. Each node reads its own local input
2. Repeat synchronous communication rounds until all nodes have announced their local outputs

Example: Find a maximal independent set \( I \)
Local output of a node \( v \) indicates whether \( v \in I \)
Synchronous distributed algorithms

- Communication round: each node
  - 1. sends a message to each port
Synchronous distributed algorithms

• Communication round: each node
  1. sends a message to each port
  (message propagation…)

![Diagram of communication round in a network with nodes and ports labeled with numbers 1, 2, and 3, showing message propagation among the nodes.]
Synchronous distributed algorithms

- Communication round: each node
  1. sends a message to each port
  2. receives a message from each port
Synchronous distributed algorithms

- Communication round:
  - 1. sends a message to each port
  - 2. receives a message from each port
  - 3. updates its own state
Synchronous distributed algorithms

- Communication round: each node
  1. sends a message to each port
  2. receives a message from each port
  3. updates its own state
  4. possibly stops and announces its output
Synchronous distributed algorithms

- Communication rounds are repeated until all nodes have stopped and announced their outputs.
- Running time = number of rounds.
- Worst-case analysis.
Synchronous distributed algorithms: networks of state machines

• Equivalently:
  • Node = state machine (not necessarily finite)
  • All nodes update their states simultaneously
Synchronous distributed algorithms: networks of state machines

- Equivalently:
  - Node = **state machine** (not necessarily finite)
  - All nodes update their states simultaneously

<table>
<thead>
<tr>
<th>Node</th>
<th>Current state</th>
<th>Port number</th>
<th>Calculation</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td></td>
<td>1</td>
<td>$a' = f_2(a, b, 2, c, 1)$</td>
</tr>
<tr>
<td>b</td>
<td></td>
<td>2</td>
<td>$b' = f_3(b, d, 1, a, 1, c, 2)$</td>
</tr>
<tr>
<td>c</td>
<td></td>
<td>3</td>
<td>$c' = f_2(c, a, 2, b, 3)$</td>
</tr>
<tr>
<td>d</td>
<td></td>
<td>1</td>
<td>$d' = f_1(d, b, 1)$</td>
</tr>
</tbody>
</table>

Current state + port number: we can reconstruct the outgoing message
Synchronous distributed algorithms: networks of state machines

- Equivalently:
  - Node = state machine (not necessarily finite)
  - All nodes update their states simultaneously

\[ a' = f_2(a, b, 2, c, 1) \]
\[ b' = f_3(b, d, 1, a, 1, c, 2) \]
\[ c' = f_2(c, a, 2, b, 3) \]
\[ d' = f_1(d, b, 1) \]
Synchronous distributed algorithms: networks of state machines

- Equivalently:
  - Node = state machine (not necessarily finite)
  - All nodes update their states simultaneously
  - Initial state = local input (incl. degree of the node)
  - Final state = local output

\[
\begin{align*}
  a' &= f_2(a, b, 2, \ldots) \\
  b' &= f_3(b, d, 1, \ldots) \\
  c' &= f_2(c, a, 2, \ldots) \\
  d' &= f_1(d, b, 1)
\end{align*}
\]
DDA 2010, lecture 1b: Computability in port-numbering model

- Impossibility of symmetry breaking
- Covering maps and covering graphs: tools for proving more impossibility results
Symmetry can’t be broken

• Input may be symmetric
  • symmetric graph
  • symmetric port numbering
  • identical local inputs
Symmetry can’t be broken

- Same input
- Same algorithm
- Same initial state
Symmetry can’t be broken

- Same current state
- Messages sent to port 1 are identical to each other
- Messages sent to port 2 are identical to each other
Symmetry can’t be broken
Symmetry can’t be broken

- Messages received from port 1 are identical to each other
- Messages received from port 2 are identical to each other
Symmetry can’t be broken

- Same old state
- Same set of received messages
- Same deterministic algorithm
- **Same new state**
Symmetry can’t be broken

- Same new state
- Either none of the nodes stops — or all of them stop and produce identical outputs
- Symmetry can’t be broken!
  - let’s formalise this...
Covering maps

\[ G = (V, E) \]

Covering map \( f: V' \rightarrow V \)

- surjection
- preserves neighbourhoods
- preserves port numbering

\[ H = (V', E') \]
Covering maps

\[ G = (V, E) \]

\[ H = (V', E') \]

Covering map \( f: V' \to V \)
- surjection
- preserves neighbourhoods
- preserves port numbering
Covering maps

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\[ G = (V, E) \]
Covering maps and covering graphs

\[ H = (V', E') \]

\[ G = (V, E) \]

\( H \) is a covering graph of \( G \) if there is a covering map \( f: V' \rightarrow V \)
Covering maps and covering graphs

- Run the **same algorithm** in $G$ and $H$
  - $v' \in V'$ and $f(v') \in V$ have the same input for all $v'$
- Then $v' \in V'$ and $f(v') \in V$:
  - have identical initial states
  - send and receive the same messages
  - have identical state transitions
  - produce **identical local outputs!**
Covering maps and covering graphs

$G = (V, E)$

$H = (V', E')$

Same output
Covering maps and covering graphs

\[ H = (V', E') \]

\[ G = (V, E) \]
Covering maps and covering graphs

\[ H = (V', E') \]

\[ G = (V, E) \]

Same output
Symmetric cycles are a simple special case of covering maps.

\[ G = (V, E) \]

\[ H = (V', E') \]
Computability in the port-numbering model

- Very limited model
  - in a cycle, we can only find a trivial solution: empty set, all nodes, ...
  - we can’t even break symmetry in a 2-node network!

- What can be solved?
DDA 2010, lecture 1c: Algorithms in port-numbering model

• Some problems *can* be solved in the port-numbering model...
  • and covering graphs can be used as an algorithm design technique, too!

• Example: vertex cover approximation
Symmetry breaking out of thin air: bipartite double covers

- Replace each node by two virtual nodes: black and white
  - original nodes simulate virtual nodes
  - each computers runs two programs in parallel: “black program” and “white program”
- Edges: black-to-white
Symmetry breaking out of thin air: bipartite double covers

- Virtual graph $H$ is a **covering graph** of $G$
- It is a **double** cover: 2 nodes of $H$ map to each node of $G$
- It is **bipartite**
  - and we have already coloured its two parts: black and white!
Symmetry breaking out of thin air: bipartite double covers
Symmetry breaking out of thin air: bipartite double covers

Port-numbering inherited
Symmetry breaking out of thin air: bipartite double covers

Port-numbering inherited
Symmetry breaking out of thin air: bipartite double covers

Port-numbering inherited
Symmetry breaking out of thin air: bipartite double covers

- Port-numbered graphs without colouring:
  - not possible to find a maximal matching (consider an even cycle)

- Port-numbered graphs with 2-colouring:
  - very easy to find a maximal matching!
Maximal matching in 2-coloured graphs

- Each white node sends proposals to its black neighbours
  - one by one, order by port numbers
Maximal matching in 2-coloured graphs

• Each white node sends proposals to its black neighbours
  • one by one, order by port numbers

• Each black node accepts the first proposal it gets
  • break ties using port numbers
Maximal matching in 2-coloured graphs

• Each white node sends proposals to its black neighbours
  • one by one, order by port numbers
  • until its proposal is accepted, or all neighbours have rejected
Maximal matching in 2-coloured graphs

• Each white node sends **proposals** to its black neighbours
  - one by one, order by port numbers

• Each black node **accepts** the first proposal it gets
  - break ties using port numbers
Maximal matching in 2-coloured graphs

• Accepted proposals $M$: matching
  • white nodes don’t propose after acceptance
  • black nodes don’t accept more than once
  • all nodes incident to at most one edge
Maximal matching in 2-coloured graphs

- Accepted proposals $M$: maximal matching!
  - assume $\{u, v\} \in E \setminus M$
    - $u$ unmatched
  - then $u$ has sent a proposal to $v$ and $v$ has rejected it
  - therefore $v$ had already received another proposal, $v$ is matched
  - can’t add $\{u, v\}$ to $M$
Maximal matching in bipartite double cover

\[ D \subseteq E \]

At least 1 of 2 virtual edges in \( M \)

\[ M \subseteq E' \]

Map back to original graph

\[ \text{At least 1 of 2 virtual edges in } M \]

\[ D \subseteq E' \]
Maximal matching in bipartite double cover

\[ D \subseteq E \]

At least 1 of 2 virtual edges in \( M \)

\[ M \subseteq E' \]

Different possibilities...
Maximal matching in bipartite double cover

\[ D \subseteq E \]

\[ M \subseteq E' \]

Different possibilities...
Maximal matching in bipartite double cover

However, this is not possible, because $M$ is a matching

- $M$ induces a subgraph of $H$ with max. degree 1

- therefore: $D$ induces a subgraph of $G$ with max. degree 2
Maximal matching in bipartite double cover

\( D \subseteq E \)

\( M \subseteq E' \)

- And this is not possible, because \( M \) is maximal
  - each edge of \( H \) is in \( M \) or shares at least one endpoint with \( M \)
  - endpoints of \( M \) form a vertex cover in \( H \)
  - endpoints of \( D \) form a vertex cover in \( G \)!
Finding a vertex cover

- So we will find a set $D$ of edges such that:
  - $D$ induces a subgraph of maximum degree 2
  - $D$ must consist of paths and cycles
  - endpoints of $D$ form a vertex cover $C$
  - is it a small vertex cover?
Finding a vertex cover

- So we will find a set $D$ of edges such that:
  - $D$ induces a subgraph of maximum degree 2
  - $D$ must consist of paths and cycles
  - endpoints of $D$ form a vertex cover $C$
  - is it a small vertex cover?

An optimal vertex cover $C^*$ needs to cover these edges, too!

Thus $C^*$ must contain 2 of these 5 nodes
Finding a vertex cover

- Different cases:
  - Cycle with 3 edges: 3 nodes in $C$, $\geq 2$ in $C^*$
  - Cycle with 4 edges: 4 nodes in $C$, $\geq 2$ in $C^*$
  - Cycle with 5 edges: 5 nodes in $C$, $\geq 3$ in $C^*$
  ...

$|C| \leq 2|C^*|$
Finding a vertex cover

- Different cases:
  - Path with 1 edge: 2 nodes in C, ≥ 1 in C*
  - Path with 2 edges: 3 nodes in C, ≥ 1 in C*
  - Path with 3 edges: 4 nodes in C, ≥ 2 in C*
  - Path with 4 edges: 5 nodes in C, ≥ 2 in C*
  ...
Finding a vertex cover

- In each path or cycle:
  - $C$ has at most 3 times as many nodes as $C^*$

- Summing over all paths and cycles:
  - $|C| \leq 3|C^*|$

- The algorithm finds a 3-approximation of minimum vertex cover!
Finding a vertex cover: summary

- Vertex cover is a graph problem that *can* be solved reasonably well in the **port-numbering model** with a deterministic distributed algorithm
  - And the algorithm was simple and fast: $O(\Delta)$ rounds!
Finding a vertex cover: two very different worlds

- Centralised setting, polynomial-time algorithms:
  - **trivial** to find a *minimal vertex cover*: greedy algorithm
  - it requires more thought to find a good *approximation of minimum vertex cover*

- Distributed setting, port-numbering model:
  - **impossible** to find a *minimal vertex cover*: symmetry breaking issues
  - but we have seen that it is possible to find a good *approximation of minimum vertex cover*
Finding a vertex cover: symmetry breaking

- Vertex cover approximation does not require symmetry breaking
  - Proof: algorithm in the port-numbering model
- However, many interesting problems do...
- Let’s study a stronger model of distributed computing