Distributed Algorithms

Algorithms for computer networks
Distributed Algorithms

Identical computers in an unknown network, all running the same algorithm
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Initially each computer only aware of its immediate neighbourhood
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Nodes can exchange messages with their neighbours to learn more...
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Finally, each computer has to stop and produce its own local output.
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Focus on graph problems:
network topology = input graph
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Focus on graph problems:
local outputs = solution  (here: graph colouring)
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Typical research question:

“How fast can we solve graph problem X?”

Time = number of communication rounds
• **Weeks 1–2:** informal introduction
  • network = path

• **Week 3:** graph theory

• **Weeks 4–7:** models of computing
  • what can be computed (efficiently)?

• **Weeks 8–11:** lower bounds
  • what cannot be computed (efficiently)?

• **Week 12:** recap
Week 1

– Warm-up: positive results
Running example: 3-colouring a path

Given a path:

Output a proper 3-colouring, e.g.:
Model of computing: 
Send, receive, update

- All nodes in parallel:
  - send messages to their neighbours
  - receive messages from neighbours
  - update their state

- Stopping state = final output
  - can send/receive, but not update any more
Challenge: Symmetry breaking

- Identical nodes, everything deterministic and synchronised: cannot break symmetry

\[ a \xrightarrow{\text{same initial state}} a \]
\[ b \xrightarrow{\text{same messages sent}} b \]
\[ \xrightarrow{\text{same messages received}} \]
\[ \xrightarrow{\text{same new state}} \]
\[ \xrightarrow{\text{...}} \]
\[ \xrightarrow{\text{same output}} \]
Challenge: Symmetry breaking

- Identical nodes, everything deterministic and synchronised: cannot break symmetry

- Solutions:
  - assume unique identifiers
  - use randomised algorithms
Algorithm P3C: Using unique IDs

- Unique IDs = proper colouring with large number of colours
- Goal: reduce the number of colours
Algorithm P3C: Using unique IDs

- Idea: **local maxima** pick a new colour
Algorithm P3C: Using unique IDs

- Idea: local maxima pick a new colour
Algorithm P3C: Using unique IDs

• Idea: local maxima pick a new colour

2 1 15 20 27 2 1 2

2 1 15 20 1 2 1 2
Algorithm P3C: Using unique IDs

- Idea: **local maxima** pick a new colour
Algorithm P3C: Using unique IDs

- Idea: *local maxima* pick a new colour
Algorithm P3C: Using unique IDs

- Inform neighbours of your current colour
- If your colour > colours of your neighbours:
  - pick a free colour from \(\{1, 2, 3\}\) that is not used by any neighbour
- Stopping states = \(\{1, 2, 3\}\)
Performance

- P3C: worst case $O(n)$
- We can do better!
Algorithm P3C Rand: Using randomness

- Initialise: state = unhappy, colour = 1
- While state = unhappy:
  - pick a new random colour from \{1, 2, 3\}
  - compare colours with neighbours
  - if different, set state = happy
Performance

• P3C: worst case $O(n)$
• P3CRand: $O(\log n)$ with high probability
• We can do better!
  • and we do not even need randomness
Algorithm P3CBit: Fast colour reduction

- Unique IDs = proper colouring with large number of colours
- Idea: reduce the number of colours from $2^k$ to $2k$ in one step
Algorithm P3CBit: Fast colour reduction

- Unique IDs = proper colouring with large number of colours
- Idea: reduce the number of colours from $2^k$ to $2k$ in one step
- Note: we will assume a directed path! (general case left as an exercise)
Algorithm P3CBit: Fast colour reduction

- Example: 128-bit unique IDs
  - \(2^{128} \rightarrow 2 \cdot 128 = 2^8 \) colours
  - \(2^8 \rightarrow 2 \cdot 8 = 2^4 \) colours
  - \(2^4 \rightarrow 2 \cdot 4 = 2^3 \) colours
  - \(2^4 \rightarrow 2 \cdot 3 = 6 \) colours

- From \(2^{128}\) to 6 colours in 4 steps! How?
Algorithm P3CBit: Fast colour reduction

$c_0$ = my current colour as a $k$-bit string
$c_1$ = successor’s colour as a $k$-bit string
$i$ = index of a bit that differs between $c_0$ and $c_1$
$b$ = value of bit $i$ in $c_0$

c = $2i + b$ = my new colour

$i \in \{0, \ldots, k - 1\}, \quad b \in \{0, 1\}, \quad c \in \{0, \ldots, 2k - 1\}$
Algorithm P3CBit: Fast colour reduction

$c_0 = 123 = 01111011_2$ (my colour)
$c_1 = 47 = 00101111_2$ (successor’s colour)
$i = 2$ (bits numbered 0, 1, 2, … from right)
$b = 0$ (in my colour bit number $i$ was 0)

$c = 2 \cdot 2 + 0 = 4$ (my new colour)

$k = 8$, reducing from $2^8 = 256$ to $2 \cdot 8 = 16$ colours
Algorithm P3CBit: Fast colour reduction

c₀ = 123 = 01111\textcolor{orange}{011}_2 \text{ (my colour)}
c₁ = 47 = 001011\textcolor{orange}{111}_2 \text{ (successor’s colour)}

Successor will pick one of these colours: 14+0, 12+0, 10+1, 8+0, 6+1, 4+1, 2+1, 0+1

None of these conflict with my choice: 4+0
Algorithm P3CBit:

Fast colour reduction

\[ i = \text{index of a bit that differs between } c_0 \text{ and } c_1 \]
\[ b = \text{value of bit } i \text{ in } c_0 \]
\[ c = 2i + b = \text{my new colour} \]

Successor picks different \( i \rightarrow \text{different } c \)
Successor picks same \( i \rightarrow \text{different } b \rightarrow \text{different } c \)

My new colour \( \neq \) my successor’s new colour
Algorithm P3CBit: Fast colour reduction

$c_0 = \text{my current colour as a } k\text{-bit string}$
$c_1 = \text{successor's colour as a } k\text{-bit string}$

$i = \text{index of a bit that differs between } c_0 \text{ and } c_1$

$b = \text{value of bit } i \text{ in } c_0$

$c = 2i + b = \text{my new colour}$

$i \in \{0, \ldots, k - 1\}, \quad b \in \{0, 1\}, \quad c \in \{0, \ldots, 2k - 1\}$
Performance

- **P3C**: worst case $O(n)$
  - assuming unique IDs
- **P3CRand**: $O(\log n)$ with high probability
- **P3CBit**: $O(\log* n)$
  - assuming unique IDs are polynomial in $n$
Performance

- **P3CBit**: $O(\log^* n)$
  - assuming unique IDs are polynomial in $n$

- **Next week**: this is optimal!
  - no deterministic distributed algorithm can 3-colour a path in time $o(\log^* n)$