• **Weeks 1–2:** informal introduction
  • network = path

• **Week 3:** graph theory

• **Weeks 4–7:** models of computing
  • what can be computed (efficiently)?

• **Weeks 8–11:** lower bounds
  • what cannot be computed (efficiently)?

• **Week 12:** recap
Week 11

– Applications of Ramsey’s theorem
Ramsey’s theorem

• For all $c, k, n$ there are numbers $R_c(n; k)$ s.t.: if we have $N \geq R_c(n; k)$ elements and we label each $k$-subset with one of $c$ colours, there is a monochromatic subset of size $n$. 
\( R_c(n; 1) \)\
\( R_c(n; 1) \leq c \cdot (n-1)+1 \)

\( R_c(n; 2) \)\
\( R_c(n; 2) \leq \tilde{R}_c(M; 2) \)

\( R_c(n; 3) \)\
\( R_c(n; 3) \leq \tilde{R}_c(M; 3) \)

\( \tilde{R}_c(n; 2) \)\
\( \tilde{R}_c(2; 2) = 2 \)

\( \tilde{R}_c(n; 2) \leq 1 + R_c(M; 1) \)

\( \tilde{R}_c(n; 3) \)\
\( \tilde{R}_c(3; 3) = 3 \)

\( \tilde{R}_c(n; 3) \leq 1 + R_c(M; 2) \)

\( M = \tilde{R}_c(2; 2) \)

\( M = \tilde{R}_c(3; 2) \leq 1 + R_c(M; 1) \)

\( M = \tilde{R}_c(3; 3) \)

\( M = \tilde{R}_c(4; 3) \leq 1 + R_c(M; 2) \)

\( \ldots \)
Applications of Ramsey’s theorem

- Application for $k = 2$, $c = 2$: any graph with $N$ nodes contains an independent set or a clique of size $n$
Applications of Ramsey’s theorem

• Application: negative results for the LOCAL model

• For any constant-time algorithm $A$, we can construct a bad input $G$ such that there is a large region of nodes with the same output
Applications of Ramsey’s theorem

• For any constant-time algorithm \( A \), we can construct a bad input \( G \) such that there is a large region of nodes with the same output
  • some technical assumptions, see exercises for details…
Applications of Ramsey’s theorem

- For any constant-time algorithm $A$, we can construct a bad input $G$ such that there is a large region of nodes with the same output
  - no constant-time algorithms for vertex colouring, edge colouring, maximal independent sets, ...
Applications of Ramsey’s theorem

- We already know all (?) this from week 2
- However, Ramsey’s theorem has further applications!
Applications of Ramsey’s theorem

• For any constant-time algorithm $A$, we can construct a bad input $G$ such that there are lots of regions of nodes with the same output
  
  • no constant-time algorithms for large independent sets, large matchings, ...
Applications of Ramsey’s theorem

- Generalisations in exercises...

- We will now just prove a simple special case: vertex colouring not possible in the LOCAL model with constant-time algorithms
Vertex colouring and Ramsey’s theorem

• **Assume:** algorithm $A$ runs in time $T = O(1)$ and outputs values 1, 2, 3

• **Claim:** there is a cycle $G$ with unique identifiers such that $A$ does not find a vertex colouring
Vertex colouring and Ramsey’s theorem

• Assume: algorithm $A$ runs in time $T = O(1)$ and outputs values 1, 2, 3

• Let: $n = 2T + 2$, $k = 2T + 1$, $c = 3$, $N = R_c(n; k)$

• Use $A$ to label $k$-subsets of $\{1, 2, \ldots, N\}$

• Monochromatic subset $\rightarrow$ bad output
Applications of Ramsey’s theorem

- $O(1)$-time algorithms cannot do much
  - even if we have unique identifiers

- $O(\log^* n)$-time algorithms much more powerful:
  - can find colourings, break symmetry, find large independent sets, …
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