Compact Representation of Sets of Binary Constraints

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Recently Rintanen, Heljanko & Niemelä (AIJ 06 or 07) have given linear size translations which help scale up to much bigger problems than earlier.

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Motivation: general problem

- A binary relation (graph) on a set of $n$ objects may have $n^2$ elements (edges).
- If the relation/graph is dense and $n$ is high ($10^4 >$) the number of elements/edges can be very high ($10^8 >$).
- The representation of the elements/edges may become impractical.
- Goal: succinct representation of the relation/graph.
motivation
cliques
explicit \(O(n^2)\) representation
\(O(n)\) representation
\(O(n \log n)\) representation
cliques vs. bicliques
application
conclusion

cliques in constraint graphs

**Definition**

Let \(\langle N, E \rangle\) be an undirected graph. Then a **clique** is \(C \subseteq N\) such that \(\{n, n'\} \in E\) for every \(n, n' \in C\) such that \(n \neq n'\).
Motivation

Cliques

Explicit $O(n^2)$ Representation

$O(n)$ Representation

$O(n \log n)$ Representation

Compression

Bicliques

Cliques vs. Bicliques

Application

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Representation with $O(n)$ Size and $O(n)$ Auxiliary Variables

[Rintanen et al. 2005]
Let $C = \{l_0, l_2, l_3, l_4, l_5, l_6, l_7\}$ be a clique consisting of 8 literals. Let $x_0, x_1, x_2$ be new Boolean variables.

\[
\begin{align*}
l_0 & \rightarrow (\neg x_0 \land \neg x_1 \land \neg x_2) \\
l_1 & \rightarrow (\neg x_0 \land \neg x_1 \land x_2) \\
l_2 & \rightarrow (\neg x_0 \land x_1 \land \neg x_2) \\
l_3 & \rightarrow (\neg x_0 \land x_1 \land x_2) \\
l_4 & \rightarrow (x_0 \land \neg x_1 \land \neg x_2) \\
l_5 & \rightarrow (x_0 \land \neg x_1 \land x_2) \\
l_6 & \rightarrow (x_0 \land x_1 \land \neg x_2) \\
l_7 & \rightarrow (x_0 \land x_1 \land x_2)
\end{align*}
\]

In general, for $n$ literals there are $n \lceil \log_2 n \rceil$ 2-literal clauses.
Complexity of finding cliques

- Finding a maximum cardinality clique is NP-hard.
- Approximation to any constant factor is NP-hard.
- Of course, polynomial-time algorithms for finding cliques exist but they have no approximation guarantees.
- (Bicliques do have polynomial-time 2-approximation algorithms!)
General compression procedure

1. Find a big clique in the constraint graph.
2. If only small cliques were found, go to the last step.
3. Represent the clique compactly.
4. Remove the edges of the clique from the constraint graph.
5. Continue from step 1.
6. Represent the remaining edges explicitly as 2-literal clauses.
**Definition**

Let $\langle N, E \rangle$ be an undirected graph. A **biclique** is a pair of $C \subseteq N$ and $C' \subseteq N$ such that $C \cap C' = \emptyset$ and $\{\{n_1, n_2\} | n_1 \in C, n_2 \in C'\} \subseteq E$.

The $nm$ edges of an $n, m$ biclique can be represented with only one auxiliary variable and $n + m$ edges.
Every clique is also a biclique
Every clique is also a biclique
Example: one 8-clique as three 4,4-bicliques
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000 → $x_0$, $x_0 \rightarrow \neg 100$
001 → $x_0$, $x_0 \rightarrow \neg 101$
010 → $x_0$, $x_0 \rightarrow \neg 110$
011 → $x_0$, $x_0 \rightarrow \neg 111$

000 → $x_1$, $x_1 \rightarrow \neg 010$
001 → $x_1$, $x_1 \rightarrow \neg 011$
100 → $x_1$, $x_1 \rightarrow \neg 110$
101 → $x_1$, $x_1 \rightarrow \neg 111$

000 → $x_2$, $x_2 \rightarrow \neg 001$
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- 011 $\rightarrow x_0$, $x_0 \rightarrow \neg 111$
- 100 $\rightarrow x_1$, $x_1 \rightarrow \neg 010$
- 101 $\rightarrow x_1$, $x_1 \rightarrow \neg 011$
- 110 $\rightarrow x_1$, $x_1 \rightarrow \neg 110$
- 111 $\rightarrow x_1$, $x_1 \rightarrow \neg 111$

- 000 $\rightarrow x_2$, $x_2 \rightarrow \neg 001$
- 010 $\rightarrow x_2$, $x_2 \rightarrow \neg 011$
- 100 $\rightarrow x_2$, $x_2 \rightarrow \neg 101$
- 110 $\rightarrow x_2$, $x_2 \rightarrow \neg 111$
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Example: one 8-clique as three 4,4-bicliques
It's equivalent to the $n \log_2 n$ encoding of cliques!

\begin{align*}
000 & \rightarrow x_0, 100 \rightarrow \neg x_0 \\
001 & \rightarrow x_0, 101 \rightarrow \neg x_0 \\
010 & \rightarrow x_0, 110 \rightarrow \neg x_0 \\
011 & \rightarrow x_0, 111 \rightarrow \neg x_0 \\
000 & \rightarrow x_1, 010 \rightarrow \neg x_1 \\
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010 & \rightarrow x_2, 011 \rightarrow \neg x_2 \\
100 & \rightarrow x_2, 101 \rightarrow \neg x_2 \\
110 & \rightarrow x_2, 111 \rightarrow \neg x_2
\end{align*}
Example: IPC Airport Problem

- Problem represents the movement of airplanes at an airport.
- Constraints on the airplane movement
- Halfway the instance series the formula sizes exceed 1 GB. Culprit: binary invariants/mutexes
- All problems this far solvable in seconds: it’s the physical size, not the actual difficulty.
Example: Compression of the Constraint Graph

Constraint graph with 62 nodes and 653 edges
Example: Compression of the Constraint Graph

Replacing $13 \times 16 = 208$ by $13 + 16 = 29$ edges.
Example: Compression of the Constraint Graph

Replacing $11 \times 18 = 198$ by $11 + 18 = 29$ edges.
Example: Compression of the Constraint Graph

Replacing $11 \times 7 = 77$ by $11 + 7 = 18$ edges.
Example: Compression of the Constraint Graph

Replacing $10 \times 7 = 70$ by $10 + 7 = 17$ edges.
Example: Compression of the Constraint Graph

Replacing $8 \times 8 = 64$ by $8 + 8 = 16$ edges.
Example: Compression of the Constraint Graph

Replacing $6 \times 6 = 36$ by $6 + 6 = 12$ edges.
Example: Compression of the Constraint Graph

Total reduction is from 653 to 121 edges.
Example: IPC Airport Problem

<table>
<thead>
<tr>
<th>instance</th>
<th>clauses for invariants before</th>
<th>clauses for invariants after</th>
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<th>size in MB after</th>
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The size reduction for many other problems is far less dramatic: 10, 30, 50 per cent.

Action mutexes / interference constraints:
- Trivial $O(n^2)$ representation (used in BLACKBOX, SatPlan, ...) catastrophic for big problems.
- We have given (Rintanen et al. 2005, 2007) linear encodings: very good scalability in comparison to BLACKBOX/SatPlan.
- Surprisingly, the biclique reduction is often better than the linear encoding, but in few cases far worse.
Other domains and applications

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  - We have given (Rintanen et al. 2005, 2007) linear encodings: very good scalability in comparison to BLACKBOX/SatPlan.
  - Surprisingly, the biclique reduction is often better than the linear encoding, but in few cases far worse.
We presented a biclique based technique for representing sets of 2-literal clauses more compactly (sometimes much more).

The basic idea is very simple and widely applicable.

Quadratic worst-case cannot be eliminated (there is a simple argument showing this.)

We have shown how compression with cliques is a special case of compression with bicliques.

Challenges: more efficient algorithms for finding big cliques and bicliques