1. Show that the energy-momentum tensor

$$\mathcal{T}_{\nu\mu} = \partial_\mu u_i \frac{\delta\mathcal{L}}{\delta(\partial^\nu u_i)} - \mathcal{L} g_{\mu\nu}$$

satisfies the continuity equation $\partial^\mu \mathcal{T}_{\mu\nu} = 0$ when the Euler-Lagrange equation is in force. The continuity equation means that something is conserved; what is it? What is the Hamiltonian density $\mathcal{H} = \mathcal{T}_{00}$ for a complex scalar field having Lagrangian density

$$\mathcal{L} = (\partial_\mu\phi)^*(\partial^\mu\phi) - m^2\phi^*\phi - \frac{1}{2}\lambda(\phi^*\phi)^2?$$

Hint: You should consider transformation properties of fields in space translations and apply Noether theorem remembering that in translations the Lagrangian does not remain invariant. For literature see e.g. Bailin and Love, chapter 3. (This should be quite a short calculation...)

2. Study the model with N real scalar fields ϕ_i, $i = 1, ..., N$, having global $O(N)$ symmetry

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi^i\partial^\mu\phi_i + \frac{1}{2}m^2\phi^i\phi_i - \frac{1}{4}\lambda(\phi^i\phi_i)^2.$$

Study the spontaneous symmetry breaking. Show that the Goldstone theorem is correct, that is, that there is a massless Goldstone boson for each generator breaking the symmetry of the vacuum. What is the residual symmetry group? Show that the generators of the residual symmetry group leave the vacuum invariant.

Hint: See Abers, Lee, Phys. Rep. 9C (1973) 14–19 (a copy can be found from the exercise box). $\frac{1}{2}(N-1)N$ generators T_{ij} of the orthogonal rotation group $O(N)$ can be written as $(T_{ij})_{kl} = -i(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$. You can start by choosing a VEV $\langle\phi\rangle$ for the field. Then you can expand the fields around the VEV as usually. There should be one massive field and $N-1$ massless fields. You should see the residual symmetry group quite easily. You should identify the generators of $O(N)$ that do not belong to the residual symmetry group, denoted by K_i. When you operate with these generators to the vacuum state you should get $K_i\langle\phi\rangle \neq 0$, i.e. they do not leave the vacuum invariant.

3. Dirac spinors are not irreducible representations of the homogeneous Lorentz group. Show this by finding an operator ($= \gamma_5$) that commutes with the generators of the homogeneous Lorentz group and the Hamiltonian for the massless Dirac spinor. Show that for massless Dirac spinor $\gamma_5\psi_D = \pm\psi_D$. Define $\psi_{DL(R)} = P_{L(R)}\psi_D$. Write the Lagrangian

$$\mathcal{L} = \frac{i}{2}\left[\overline{\gamma}_D\gamma^\mu\partial_\mu\psi_D - (\partial_\mu\overline{\gamma}_D)\gamma^\mu\psi_D\right] - m_D\overline{\gamma}_D\psi_D$$

in terms of its chiral components $\psi_{DL(R)}$ ($m_D \neq 0$) and show that at massless limit the chiral components decouple and that there are separate equations of

motion for both of them.
Hint: $P_L = \frac{1}{2}(1-\gamma_5)$, $P_R = \frac{1}{2}(1+\gamma_5)$. For literature see any textbook on advanced relativistic quantum mechanics. (E.g. Itzykson-Zuber, Halzen-Martin, ...)

Return the solved problems to the downstairs box in Laurikainen building on Monday October 5th at 12 o'clock at latest. The www-page of this course is:
http://www.physics.helsinki.fi/%7Epuolamak/hitu/