1. Obtain the Euler-Lagrange equations of motion for theories whose Lagrange densities are

   (i) $\mathcal{L}_0(\varphi, \partial_\mu\varphi) = \frac{1}{2}[\partial^\mu\varphi(x)\partial_\mu\varphi(x) - M^2\varphi^2(x)]$

   (ii) $\mathcal{L}_0(\varphi, \partial_\mu\varphi) = \partial^\mu\varphi^*(x)\partial_\mu\varphi(x) - M^2\varphi^*(x)\varphi(x)$

   (iii) $\mathcal{L}_0(A_\mu, \partial_\mu A_\nu) = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$

   (iv) $\mathcal{L}_0(\psi, \partial_\mu\psi) = \frac{i}{2}[\bar\psi(x)\partial\!\!\!/\psi(x) - (\partial^\mu\bar\psi(x))\gamma_\mu\psi(x)] - m\bar\psi(x)\psi(x)$.

2. Show that the Lagrangians

$$\mathcal{L}_0(\varphi, \partial_\mu\varphi) = \partial^\mu\varphi^*(x)\partial_\mu\varphi(x) - m^2\varphi^*(x)\varphi(x)$$

   and

$$\mathcal{L}_0(\psi, \partial_\mu\psi) = \frac{i}{2}[\bar\psi(x)\partial\!\!\!/\psi(x) - (\partial^\mu\bar\psi(x))\gamma_\mu\psi(x)] - M\bar\psi(x)\psi(x),$$

   where $\psi = (\psi_1, \psi_2)^T$, are invariant in global infinitesimal SU(2)-transformations

$$\varphi(x) \to \varphi'(x) = \varphi(x), \psi^a(x) \to \psi'^a(x) = (\delta^{ab} - \frac{i}{2}\varepsilon^k {\tau_k}^{ab})\psi_b(x),$$

   i.e. $\varphi(x)$ is isosinglet and $\psi(x)$ isodoublet.
   Hint: $\bar\psi = (\bar\psi_1, \bar\psi_2)$ and $A\!\!\!/ \equiv A_\mu\gamma^\mu$.

3. What are the conserved currents in global U(1)-transformations for the Lagrangian

$$\begin{aligned}\mathcal{L} &= \frac{i}{2}[\bar\psi(x)\partial\!\!\!/\psi(x) - \partial^\mu\bar\psi(x)\gamma_\mu\psi(x)] - M\bar\psi(x)\psi(x) \\ &\quad -e\bar\psi(x)\gamma_\mu\psi(x)A^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}.\end{aligned}$$

   How would you interpret the result?

4. (Corrected version, clarified hints and notation.)

$$\mathcal{L} = \overline{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_a^{\mu\nu}F^a_{\mu\nu},$$

   where

$$D^\mu = \partial^\mu + igT^aA_a^\mu,$$
$$F_a^{\mu\nu} = \partial^\mu A_a^\nu - \partial^\nu A_a^\mu - gf_{abc}A_b^\mu A_c^\nu.$$

   T_a are generators of some simple Lie group obeying commutation relation $[T_a, T_b] = if_{abc}T_c$.

   Show that the Lagrangian is invariant in *finite* local (=gauge) transformations

$$\psi(x) \to U(x)\psi(x),$$

$$A^\mu(x) \to U(x)(A^\mu(x) - \frac{i}{g}\partial^\mu)U^{-1}(x) = U(x)A^\mu(x)U^{-1}(x) - \frac{i}{g}U(x)(\partial^\mu U^{-1}(x)),$$

where $U(x) = e^{-igT^a\Lambda_a(x)}$ and $A_\mu(x) = A_\mu^a(x)T_a$.

Hint: The first part of the Lagrangian and the Yang-Mills Lagrangian are separately gauge invariant. You can begin by showing that the covariant derivative transforms as $D^\mu(x)\psi(x) \to U(x)D^\mu(x)\psi(x)$, as it should. You can use this to show that the first part of the Lagrangian is gauge invariant. The Yang-Mills Lagrangian can be written in matrix form using $F^{\mu\nu} = F_a^{\mu\nu}T^a (= -\frac{i}{g}[D^\mu, D^\nu])$ and the normalization property of the simple Lie groups $\mathrm{Tr}T_kT_l = 2\delta_{kl}$. (If you want, you can take the simple Lie group to be SU(2); then $T_a = \tau_a$, $a = 1, 2, 3$, and $f_{abc} = 2\epsilon_{abc}$. This is not necessary, though.) For helpful literature on the topic see e.g. Bailin and Love, chapter 9 "Gauge field theories".

Return the solved problems to the downstairs box in Laurikainen building on Monday September 28th at 12 o'clock at latest. The first problem session is on Tuesday 16-18 in the Exercise room (harjoitussali), Laurikainen building, [English] and the second is on Friday 10-12 in Hall 5, Tallqvist building [Finnish].

The www-page of this course is: http://www.physics.helsinki.fi/%7Epuolamak/hitu/