Theories of particle physics — Fall 1998, Exercise 1

1. Find the solutions to the Klein-Gordon equation
$$(\partial^\mu \partial_\mu + m^2)\psi(x) = 0.$$

2. Find the solutions to the Dirac equation
$$(i\partial_\mu \gamma^\mu - m)\psi(x) = 0,$$
and interpret them physically (see e.g. Björken-Drell, Merzbacher, Itzykson-Zuber, Halzen-Martin or any other textbook in advanced quantum physics). *Total* angular momentum should be conserved, that is, it should commute with the Hamiltonian for Dirac spinor. What is the total angular momentum operator for Dirac spinor (see e.g. Halzen and Martin, especially exercise 5.4)?

3. A representation for a linear algebra, called Clifford algebra, can be constructed from the matrices γ_μ in the following way:
$$\Gamma^S = 1,\ \Gamma^V_\mu = \gamma_\mu,\ \Gamma^T_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu],\ \Gamma^A_\mu = \gamma_5\gamma_\mu,\ \Gamma^P = \gamma_5.$$
Show that this set $\{\Gamma^a\}_{a=1}^{16}$ is complete, i.e. it generates the 16 dimensional vector space of the linear algebra. Hint: You should show

   (a) $(\Gamma^a)^2 = \pm 1$,
   (b) $\forall \Gamma^a \neq 1\ \exists \Gamma^b$ s.e. $\Gamma^a\Gamma^b = -\Gamma^b\Gamma^a$,
   (c) so that $\mathrm{Tr}(\Gamma^b) = 0,\ \Gamma^b \neq 1, a \neq b$,
   (d) $\Gamma^a\Gamma^b = d_{abc}\Gamma^c,\ d^4_{abc} = 1,\ \Gamma^c \neq 1,\ a \neq b$
   (e) so that $\lambda_a\Gamma^a = 0 \Rightarrow \lambda_a = 0$.

   Show the above for as many a,b that it is sufficiently believable.(Itzykson-Zuber, Björken-Drell)

4. Show in detail that the bilinear covariants
$$\bar\psi(x)\psi(x),\quad \bar\psi(x)\gamma_5\psi(x),\quad \text{and}\quad \bar\psi(x)\gamma_\mu\psi(x)$$
of the spinor $\psi(x)$ transform in the Lorentz transformations Λ as a scalar, pseudoscalar and vector.
Hint: You should show ${\Lambda^\nu}_\alpha {\Lambda_\mu}^\alpha = {g^\nu}_\mu$, $\gamma^0 S^\dagger(\Lambda)\gamma^0 = S^{-1}(\Lambda)$, $S^{-1}(\Lambda)\gamma_5 S(\Lambda) = \det(\Lambda)\gamma_5$. Also note that the matrix $S(\Lambda)$ can be defined requiring the covariance of the Dirac equation. (See e.g. Björken-Drell I or Schweber for Lorentz transformations and inversions of space)

General advice: Avoid writing the gamma matrices and spinors in component form as much as possible - use the general properties of the gamma matrices as far as you can! You might find Juha's appendix for the last year's course quite useful in dealing with gamma matrices (see the web page).

Return the solved exercises to the downstairs box in Laurikainen building on Monday September 21st at 12 o'clock at latest. The first problem session is on Tuesday 16-18 [English] (as yet I have no idea where, you can read it from the web page or ask me later) and the second is on Friday 10-12 in Hall 5, Tallqvist building [Finnish].

The www-page of this course is: http://www.physics.helsinki.fi/%7Epuolamak/hitu/