

# Systems and Signals 2013/14

## Course Notes. Part I.

Jesse Read



Version: November 26, 2013

# 1 Course Contents and Resources

## Timetable

- Lecture: Wednesday 11:00 to 13:00, room 7.1.J06
- Seminar: Friday 11:00 to 13:00 , room 7.1.J06
  - *except* labs: 18 October, 15 November, 29 November, room INF 4.S.D.01
  - and *except* Thursday 10 October 15:00–17:00 – (replaces Thursday 11 lecture), room 7.1.J02
- Lab Exam: Friday 13 December 13:00 to 15:00 in INF 4.S.D.01

## Assessment

- 60 % final exam (6 points)
- 40 % quizzes, lab exam session (4 points)
  - 3 quizzes, 1 point each
  - 1 lab exam, 1 point

## Book

- (COURSE BOOK) Alan V. Oppenheim, Alan S. Willsky, S. Hamid Nawab. *Signals and Systems*. 2nd Ed. 1997.

## Other Resources

- My notes (I will put them on AULA GLOBAL after lectures, prior to homework exercises)
- MIT lecture videos and slides  
<http://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-003-signals-and-systems-fall-2011/lecture-videos-and-slides/>
- Instructional YouTube videos  
<https://sites.google.com/a/asu.edu/signals-and-systems/>
- Another book:  
Charles L. Phillips. John M. Parr. *Signals, Systems, and Transforms*.

**Course Contents** The detailed course plan is already / will shortly be uploaded to AULA GLOBAL. There is already information about course content available in AULA GLOBAL.

1. Part 1: Jesse Read
  - Contact hours flexible – but recommended to check by e-mail first )
  - By email: [jesse@tsc.uc3m.es](mailto:jesse@tsc.uc3m.es)
  - Laboratorio 4.3.A.03
2. Part 2: Jose Miguel Leiva Murillo (Course Coordinator)

# Contents

<b>1</b>	<b>Course Contents and Resources</b>	
<b>2</b>	<b>Signals</b>	<b>1</b>
2.1	Definition . . . . .	2
2.2	Examples of signals . . . . .	2
2.3	Properties of signals . . . . .	3
2.3.1	Periodicity . . . . .	3
2.3.2	Symmetry . . . . .	6
2.4	Signal Characterization, Power and Energy . . . . .	7
2.4.1	Signal Power . . . . .	7
2.4.2	Signal Energy . . . . .	8
2.5	Transformations of the independent variable . . . . .	10
2.6	Operations on signals . . . . .	12
2.7	Basic/Elementary Signals . . . . .	14
2.7.1	Unit impulse and unit step . . . . .	14
2.7.2	Exponential and sinusoidal signals . . . . .	17
2.8	Signals as Vectors . . . . .	22
<b>3</b>	<b>Systems</b>	<b>25</b>
3.1	Introduction . . . . .	26
3.2	Interconnection of Systems . . . . .	29
3.2.1	Series (cascade) . . . . .	29
3.2.2	Parallel . . . . .	29
3.2.3	Combination . . . . .	30
3.2.4	Feedback . . . . .	32
3.3	Properties . . . . .	37
3.3.1	Memory / Memoryless . . . . .	37
3.3.2	Invertibility . . . . .	38
3.3.3	Causality . . . . .	38
3.3.4	Stability . . . . .	39
3.3.5	Time invariance . . . . .	40
3.3.6	Linearity . . . . .	41
3.4	Linear Time-Invariant Systems (LTIS) . . . . .	44
3.4.1	Linearity . . . . .	44
3.4.2	Impulse response . . . . .	45
3.5	Convolution . . . . .	46
3.6	Interconnection of LTI Systems . . . . .	57
3.7	(Unit) step response to an LTIS . . . . .	58
<b>4</b>	<b>Fourier Series</b>	<b>60</b>
4.1	Response of LTI Systems to Complex Exponentials . . . . .	61
4.2	Introduction to Fourier Series and Frequency Analysis . . . . .	63
4.3	Fourier Series Representation of CT Periodic Signals: Analysis and Synthesis Equations . . . . .	64
4.4	Convergence . . . . .	71
4.5	Properties of the CT Fourier Series . . . . .	72
4.6	FS Representations of DT Periodic Signals . . . . .	73
4.7	Properties of DT FS and comparison with CT FS . . . . .	79

<b>A</b>	<b>Appendix: Math Review</b>	<b>81</b>
A.1	Complex Numbers . . . . .	82
A.2	Euler's formula . . . . .	84
A.3	Geometric Series . . . . .	86
A.4	Taylor Series Expansion . . . . .	87

## 2 Signals

## 2.1 Definition

A signal ...

- ... is a description of a physical phenomenon
- ... is a time/space representation of information
- ... describes the time/space variation of a magnitude
- (formal) is a function of one or more independent variables.

## 2.2 Examples of signals

- Voice  $v(t)$
- Radio  $r(t)$
- B&W Photographs  $i[x, y]$  (intensity of pixels)
- Electrocardiograms (ECG)  $e(t)$
- Stock market  $s[n]$
- Voltage, air pressure, ...

We will generally be interested in functions of time  $t \in \mathbb{R}$  (continuous time – CT) or sequence  $n \in \mathbb{Z}$  (discrete time – DT). The signals is  $\in \mathbb{R}$  (real space), or  $\in \mathbb{C}$  (complex space).

Real:

$$x(t) : \mathbb{R} \rightarrow \mathbb{R}$$
$$x(t) = e^{-\alpha t} + \beta t$$

$$x[n] : \mathbb{Z} \rightarrow \mathbb{R}$$
$$x[n] = \alpha^{|n|}$$

or Complex:

$$x(t) : \mathbb{R} \rightarrow \mathbb{C}$$
$$x(t) = Ae^{j(\omega_0 t + \phi)}$$

$$x[n] : \mathbb{Z} \rightarrow \mathbb{C}$$
$$x[n] = \frac{4n + j8n}{n^2 + j4n^3}$$

Signals may be

- deterministic
- probabilistic

## 2.3 Properties of signals

### 2.3.1 Periodicity

A continuous **periodic** signal:

$$x(t) = x(t + T)$$

for *all*  $t$ .

The **fundamental period**  $T_0$  is the *minimum*  $T$  for which periodicity can be shown. Other periods are  $T_0k$  for any  $k = 1, 2, 3, \dots$

The **radial frequency** (in radians per second) is

$$\omega_0 = \frac{2\pi}{T_0} \quad (1)$$

and from this we obtain

$$T_0 = \frac{2\pi}{\omega_0} \quad (2)$$

For example, if we consider  $\cos(t)$  (where implicitly  $\omega_0 = 1$ ), we can see that the fundamental period is

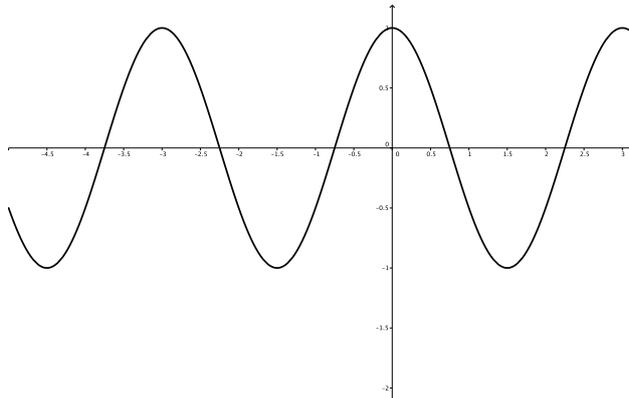
$$T_0 = \frac{2\pi}{1} = 2\pi$$

**Example:**

$$x(t) = \cos\left(\frac{2}{3}\pi t\right)$$

Is this periodic? What is the period?

$$x(t + T) = \cos\left(\frac{2}{3}\pi(t + T)\right) = x(t) ?$$



Yes, it's periodic, for which period?

We identify  $\omega_0 = \frac{2}{3}\pi$  (see Eq. 6 in Section 2.7.2) and from Eq. 2 we get

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi \frac{3}{2\pi} = 3$$

(as we can also see in the plot).

Example:

$$x(t) = \cos(2\pi t) + \sin\left(\frac{\pi}{3}t\right)$$

Is  $x(t)$  periodic?

$$T_1 = \frac{2\pi}{\omega_1} = 2\pi \frac{1}{2\pi} = 1$$

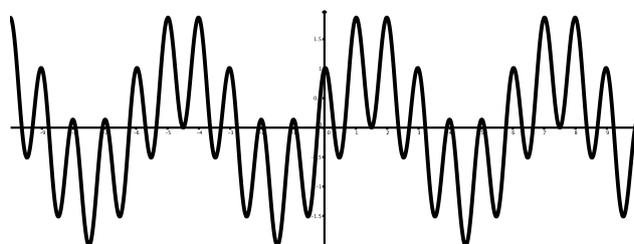
$$T_2 = \frac{2\pi}{\omega_2} = 2\pi \frac{3}{\pi} = 6$$

We need some  $k_1$  and  $k_2$  such that

$$k_1 T_1 = k_2 T_2$$

i.e., the **least common multiple** (LCM), which in this case = 6. Thus, fundamental period is

$$T_0 = 6$$



A discrete signal  $x[n]$  is periodic if there exists an *integer number*  $N$  such that

$$x[n] = x[n + N]$$

for any  $n$ , which requires that

$$\omega_0 N = 2\pi k$$

for  $k$  an *integer number*.

Example: Is the following periodic?

$$x[n] = \cos\left(\frac{1}{7}\pi n\right)$$

The fundamental period is

$$N_0 = \frac{2\pi}{\omega_0} = \frac{2\pi}{\frac{1}{7}\pi} = 2\pi \frac{7}{\pi} = 14$$

which is an integer, so  $x[n]$  is periodic. The signal repeats every  $Nk = 14k$  times for  $k = 1, 2, 3, \dots$

Example: Is the following periodic?

$$x[n] = \cos\left(\frac{1}{7}n\right)$$

We look for

$$N = \frac{2\pi}{\omega_0} = \frac{2\pi}{\frac{1}{7}} = 2\pi \cdot 7 = 14\pi$$

but we note that this is not an integer number. We can try to get an integer with

$$Nk = 14\pi k$$

trying  $k = 1, 2, 3, 4, \dots$  but there is no  $k$  which gives us an integer  $N$ ! Thus,  $x[n]$  is *not* periodic (although it may look periodic if we plot it).

Example: Is the signal

$$x[n] = e^{j\frac{3\pi}{5}n+1}$$

periodic? We look for

$$N = \frac{2\pi}{\omega_0} = \frac{2\pi}{3\pi/5} = \frac{10}{3}$$

This is not an integer number, but can try to get an integer with

$$Nk = \frac{10}{3}k$$

for integers  $k = 1, 2, 3, 4, \dots$  and we find that with  $k = 3$  we get an integer

$$N \cdot 3 = 10$$

Therefore, this signal *is* periodic (with fundamental period  $N_0 = 10$ ).

Example: What is the period of the discrete signal from Fig. 1?

To be periodic:

$$\cos\left(\frac{\pi}{10}n^2\right) = \cos\left(\frac{\pi}{10}(n+N)^2\right) = \cos\left(\frac{\pi}{10}n^2 + \underbrace{\left[\frac{\pi}{10}N^2 + \frac{\pi}{10} \cdot 2nN\right]}_{\text{solve for } N}\right)$$

$$\frac{\pi}{10}N^2 + \frac{\pi}{10}2nN = k2\pi$$

$$2\pi\left(\frac{N^2}{20} + \frac{nN}{10}\right) = k2\pi$$

$$\frac{N^2}{20} + \frac{n}{10}2nN = k$$

where  $k$  is an integer for (minimum)  $N_0 = 20$ !

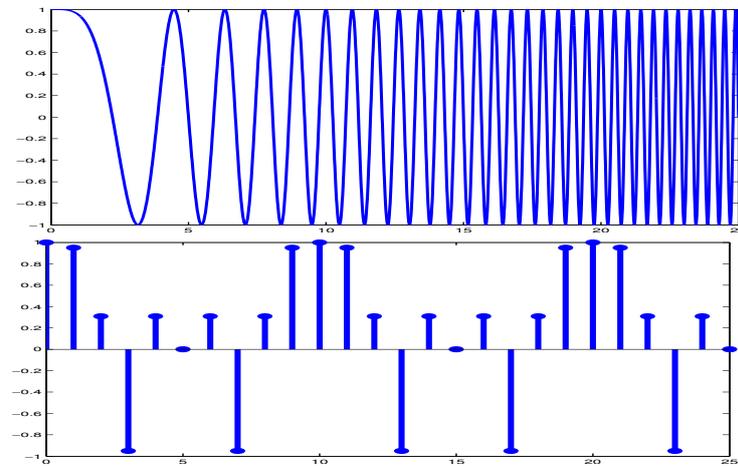


Figure 1: An example of why our intuition with CT signals sometimes does not cross over to DT signals. The top graph shows  $x(t) = \cos(\frac{\pi}{10}t^2)$  (clearly non periodic), the bottom graph shows  $x[n] = \cos(\frac{\pi}{10}n^2)$  (periodic).

### 2.3.2 Symmetry

Is a signal **even** or **odd** symmetrical?

<b>even</b> signal	$x(-t) = x(t)$	(reflection around 0)
<b>odd</b> signal	$x(-t) = -x(t)$	(upside-down reflection)

#### Important fact

Any signal can be broken into a sum of two signals: even and odd.

The signal  $x(t)$  is broken into even

$$x_{\text{even}}(t) = \frac{1}{2}[x(t) + x(-t)]$$

and odd parts

$$x_{\text{odd}}(t) = \frac{1}{2}[x(t) - x(-t)]$$

**Example:**

So,

$$x_{\text{even}}(-3) = \frac{1}{2}[x(-3) + x(3)] = \frac{1}{2}[0 + 1] = 0.5$$

and so on, and

$$x_{\text{odd}}(-3) = \frac{1}{2}[x(-3) - x(3)] = \frac{1}{2}[0 - 1] = -0.5$$

and so on.

Useful fact: **sine** is an **odd function**, **cosine** is an **even function**:

<b>even</b>	$\cos(-t) = \cos(t)$	(can 'fold' along vertical axis)
<b>odd</b>	$\sin(-t) = -\sin(t)$	(can 'spin' around the origin)

## 2.4 Signal Characterization, Power and Energy

The **average value** of a signal, over period  $T$  is (for any  $t_0$ ):

$$\langle x(t) \rangle = \frac{1}{T} \int_{t_0-T/2}^{t_0+T/2} x(t) dt$$

The **total average value** is:

$$\langle x(t) \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t_0-T/2}^{t_0+T/2} x(t) dt$$

**Peak value**

$$x_p = \max_t |x(t)|$$

### 2.4.1 Signal Power

A signal could capture physical phenomena like **power**. The **instantaneous power** at time  $t$  is

$$P(t) = |x(t)|^2$$

(and  $P[n] = |x[n]|^2$  for DT signals) where  $|x(t)|$  is the **magnitude** of the [possibly complex] value at time  $t$ . Note that  $P(t)$  is a signal!

The **average power** of signal  $x(t)$  over a given interval of time  $T$  around some point  $t_0$  is

$$P = \frac{1}{T} \int_{t_0-\frac{T}{2}}^{t_0+\frac{T}{2}} |x(t)|^2 dt$$

For periodical signals, it makes sense to work with the fundamental period  $T_0$ . The **total average power** (over an infinite interval of time) is :

$$P_\infty = \lim_{T \rightarrow \infty} \left\{ \frac{1}{T} \int_{t_0 - \frac{T}{2}}^{t_0 + \frac{T}{2}} |x(t)|^2 dt \right\}$$

and

$$P_\infty = \lim_{N \rightarrow \infty} \left\{ \frac{1}{2N + 1} \sum_{n=n_0-N}^{n_0+N} |x[n]|^2 \right\}$$

for CT and DT, respectively. . For periodical signals, this is equivalent to  $T = T_0$  (the fundamental period).

### 2.4.2 Signal Energy

The **energy** of a signal in the *interval* between  $t_0$  to  $t_1$  is

$$\begin{aligned} E &= \int_{t_0}^{t_1} |x(t)|^2 dt \\ &= \int_{t_0}^{t_1} P(t) dt \end{aligned}$$

The **total energy** of a signal  $x(t)$  is

$$\begin{aligned} E_\infty &= \int_{-\infty}^{\infty} |x(t)|^2 dt \\ &= \lim_{T \rightarrow \infty} \left\{ \int_{t_0 - \frac{T}{2}}^{t_0 + \frac{T}{2}} |x(t)|^2 dt \right\} \end{aligned}$$

and, in the discrete domain:

$$\begin{aligned} E_\infty &= \sum_{n=-\infty}^{\infty} |x[n]|^2 \\ &= \lim_{N \rightarrow \infty} \left\{ \frac{1}{2N + 1} \sum_{n=-N}^N |x[n]|^2 \right\} \end{aligned}$$

i.e., the area under the instantaneous *power curve* over all time. .

### Infinite power/energy

- $P_\infty < \infty$  finite total average power
- $E_\infty < \infty$  finite total energy

So we can have signals with

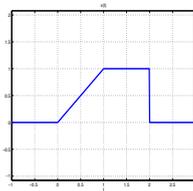
- *infinite* total energy but *finite* total average power (for example,  $x(t) = \sin(t)$ )
- *infinite* total energy and *infinite* power is (for example,  $x(t) = t$ )
- *finite* total energy and *finite* power (for example,

$$x(t) = \begin{cases} 1 & 0 \leq t \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

)

### Example:

$$x(t) = \begin{cases} t & 0 \leq t \leq 1 \\ 1 & 1 \leq t \leq 2 \\ 0 & \text{otherwise} \end{cases}$$



What is the **total energy** of this signal?

$$\begin{aligned} E_\infty &= \int_0^1 |x(t)|^2 dt + \int_1^2 |x(t)|^2 dt \\ &= \int_0^1 t^2 dt + \int_1^2 1^2 dt \\ &= \frac{1}{3} [t^3]_0^1 + [1]_1^2 dt \\ &= \left[ \frac{1}{3} 1^3 - \frac{1}{3} 0^3 \right] + [2 - 1] \\ &= \frac{1}{3} + 1 \\ &= 1\frac{1}{3} \end{aligned}$$

The **root mean square** value is the root square of the average power of a signal over an interval:

$$x_{\text{RMS}} = \sqrt{\frac{1}{|t_2 - t_1|} \int_{t_1}^{t_2} |x(t)|^2 dt}$$

and, of course, there is a discrete version:

$$x_{\text{RMS}} = \sqrt{\frac{1}{|n_2 - n_1|} \sum_{n_1}^{n_2} |x[n]|^2}$$

## 2.5 Transformations of the independent variable

- Time reversal

$$y(t) = x(-t)$$

- Time shifting

$$y(t) = x(t - t_d)$$

if  $t_d > 0$  then delayed, if  $t_d < 0$  then forwarded

- Time scaling

$$y(t) = x(\lambda t)$$

where  $\lambda < 1$  for expansion,  $\lambda > 1$  for compression,  $\lambda = -1$  for reversal

The same goes for discrete signals:

$$y[n] = x[-n]$$

and so on. In other contexts the independent variable could be other than time.

Operations can be combined. For example

$$y(t) = x(2t - 1)$$

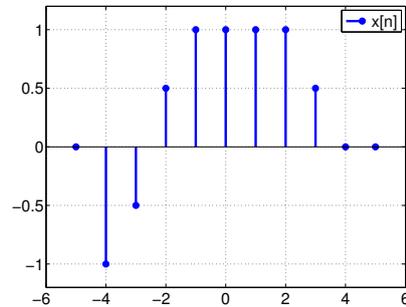
involves a scale and a shift.

$$\begin{aligned} y(t) &= x(-t - 1) \\ &= x(1 - t) \end{aligned}$$

involves a time reversal and a shift.

However, in many cases it is safer to figure out the result graphically.

Example: We have the signal



upon which we wish to do the transformation:

$$y[n] = x[3n + 1]$$

This means that the point at  $n = 3$  in  $y[n]$  is located at  $n = 3 \cdot 3 + 1 = 7$  in  $x[n]$ . This can be difficult to visualise when it comes to graphing  $y[n]$ . We can switch the equation  $n_x = 3n_y + 1$ , solving for  $n_y$ ; and thus:

$$x[n] = y\left[\frac{n-1}{3}\right]$$

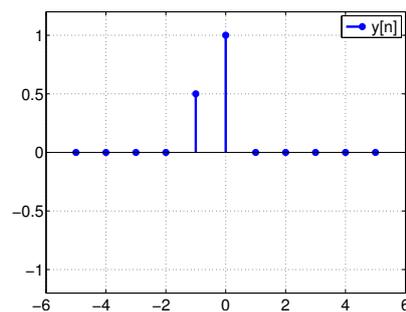
meaning that the point  $n = 7$  in  $x[n]$  is located at  $n = \frac{7-1}{3} = 2$  in  $y[n]$ . This means we can just plug in the interesting  $n$ -indices in  $x[n]$  and get the result. In MATLAB we could just do

$$\text{n\_x} = [-4:1:3]$$

and then

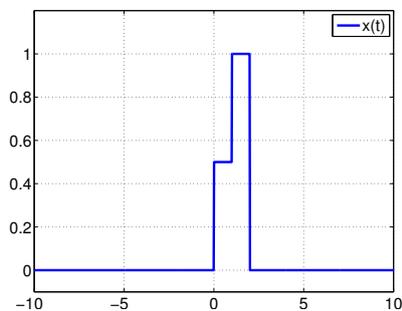
$$\text{n\_y} = (\text{n}-1)/3$$

Or, by hand, we start transferring points from the graph at  $n$  to  $n-1/3$  until we see a pattern emerge, and thus until:



(Recall that only integer numbers  $n$  exist in a DT signal).

Example: We have the signal

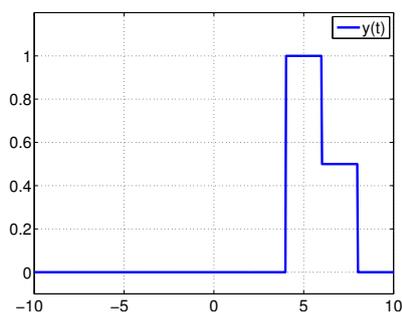


$$y(t) = x\left(4 - \frac{t}{2}\right)$$

This operation involves a time reversal, scale, and shift. As in the previous example, we can reverse it:

$$x(t) = y(-2t + 8)$$

We start transferring points from the graph at  $t$  to  $-2t + 8$  until we see a pattern emerge, and thus until:



A good set of animations is provided at <http://www.ee.washington.edu/class/235d1/EE235/Project/lesson2/lesson2.html>

## 2.6 Operations on signals

- Amplitude scaling

$$y(t) = \lambda x(t)$$

- Addition

$$y(t) = x_1(t) + x_2(t)$$

- Multiplication

$$y(t) = x_1(t) \cdot x_2(t)$$

- Modulo

$$y(t) = |x(t)|$$

- Derivative

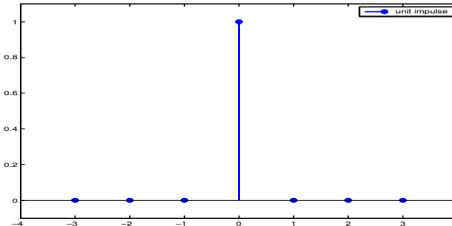
$$y(t) = \frac{d}{dt}x(t)$$

## 2.7 Basic/Elementary Signals

### 2.7.1 Unit impulse and unit step

The most simple signal is the **unit impulse** response / function,  $\delta[n]$ , i.e., the “Kronecker delta”

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$



which we can displace by  $n_0$ ,

$$\delta[n - n_0] = \begin{cases} 1 & n = n_0 \\ 0 & n \neq n_0 \end{cases}$$

We can multiply a signal by  $\delta[n]$

$$x[n] \cdot \delta[n]$$

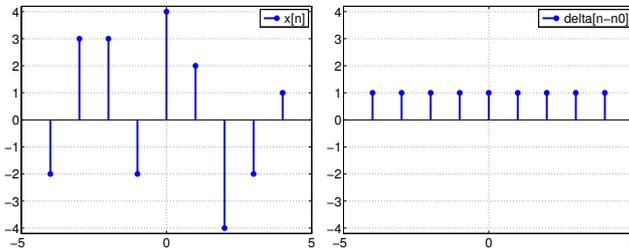
or by a shifted  $\delta[n - n_0]$ ,

$$x[n] \cdot \delta[n - n_0]$$

In fact, any sequence can be described by a combination of offset unit impulses! Thus,

$$x[n] = \sum_{k=-\infty}^{+\infty} \underbrace{x[k]}_{\text{constant}} \underbrace{\delta[n - k]}_{\text{signal}} \quad (3)$$

Example: The signal  $x[n]$  composed from scaled offset unit impulses  $\delta[n - n_0]$  (for  $n_0 = \dots, -3, -2, -1, 0, 1, 2, 3, \dots$ ):



$$x[n] = \dots + 3\delta[n + 2] + -2\delta[n + 1] + 4\delta[n] + 2\delta[n - 1] + 4\delta[n - 2] \dots$$

$$= \dots + x[-2]\delta[n + 2] + x[-1]\delta[n + 1] + x[0]\delta[n] + x[1]\delta[n - 1] + x[2]\delta[n - 2] \dots$$

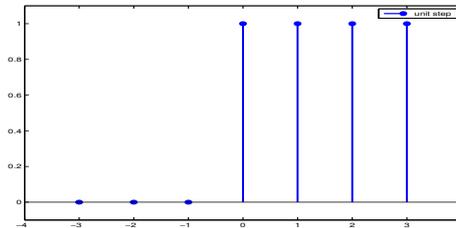
The running sum of the delta over  $k$  is the **unit step** sequence / function,

$$u[n] = \dots + \delta[-3] + \delta[-2] + \dots + \delta[n]$$

$$= \sum_{k=-\infty}^n \delta[k]$$

where  $u[n] = 1$  for  $n = \infty$ , and generally:

$$u[n] = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0 \end{cases}$$

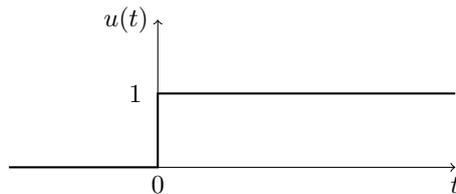


We could also express the unit sample  $\delta[n]$  as the first backward difference of the unit step  $u[n]$ :

$$\delta[n] = u[n] - u[n - 1]$$

The continuous time **unit step**  $u(t)$ , is similar to the discrete case:

$$u[n] = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

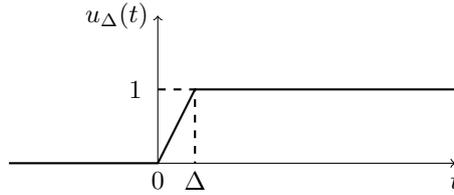


But this function is discontinuous at 0! We instead approximate  $u_{\Delta}(t)$ , as

$$u[n] = \begin{cases} 0 & t < 0 \\ t/\Delta & 0 < t < \Delta \\ 1 & t \geq \Delta \end{cases}$$

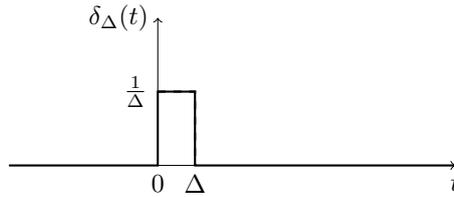
for very small  $\Delta$ . Therefore, we have

$$u(t) = \lim_{\Delta \rightarrow 0} u_{\Delta}(t)$$



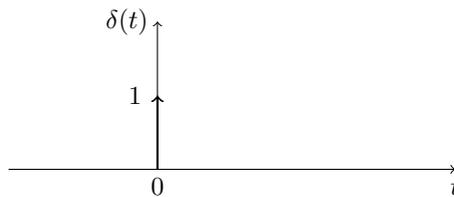
with derivative:

$$\delta_{\Delta}(t) = \frac{d}{dt}u_{\Delta}(t) = \begin{cases} \frac{1}{\Delta} & 0 \leq t \leq \Delta \\ 0 & \text{otherwise} \end{cases} \quad (4)$$



which, for very small  $\Delta$ , is the continuous Kronecker delta  $\delta(t)$  :

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$



where the 1 represents the area. As in the DT case, we can get back to the unit step as follows (an integral instead of a sum) :

$$u(t) = \int_{-\infty}^t \delta(\tau) d\tau$$

where  $u(t) = 1$  for  $t > 0$  (i.e., also when  $t = \infty$ ).

The unit step is often used to switch functions, for example

$$\cos \omega t \cdot u(t) = \begin{cases} \cos \omega t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

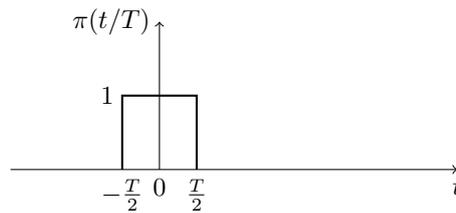
$$e^{at}u(t) = \begin{cases} e^{at} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Note also, the **rectangular pulse**

$$\pi(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

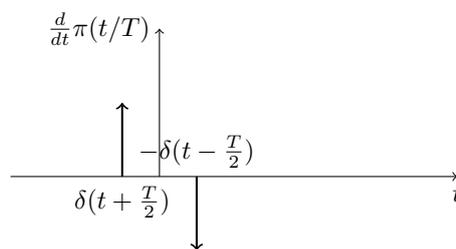
or, generally,

$$\pi\left(\frac{t}{T}\right) = \begin{cases} 1 & |t| < \frac{T}{2} \\ 0 & \text{otherwise} \end{cases}$$



And the relation to  $\delta(t)$  is via the derivative,

$$x(t) = \frac{d}{dt}\pi\left(\frac{t}{T}\right)$$



and to get back again, take the integral

$$\pi(t) = \int_{-\infty}^t x(\tau) d\tau$$

### 2.7.2 Exponential and sinusoidal signals

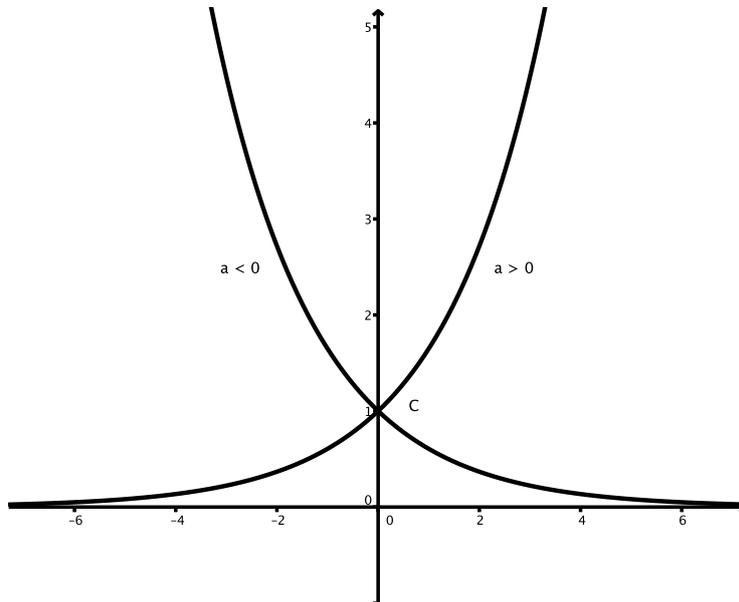
The continuous **exponential signal** is

**Continuous exponential**

$$x(t) = Ce^{at}$$

with

- (real constant)  $C \in \mathbb{R}$ , and
- $a \in \mathbb{R}$



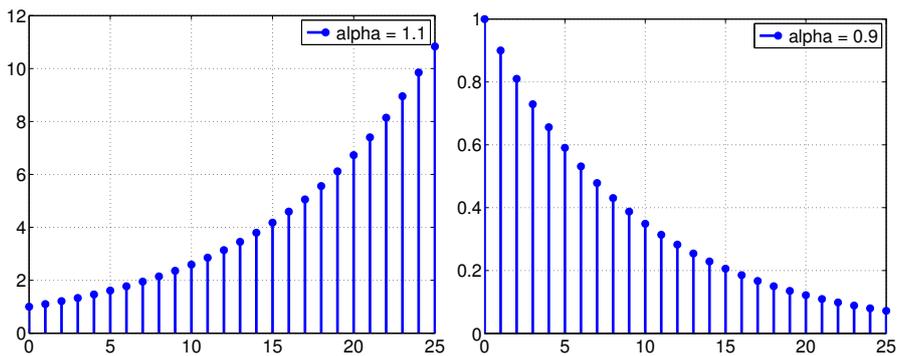
where  $a > 0$  results in exponential growth , and  $a < 0$  results in exponential decay . This is **aperiodic!**

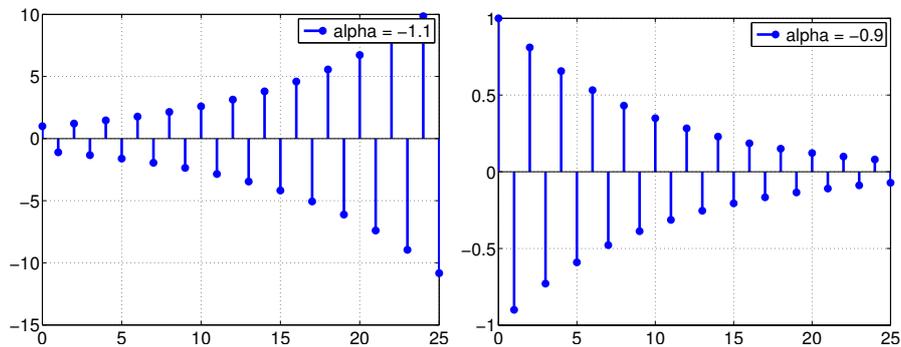
The *discrete* exponential signal is:

**Discrete Exponential signal**

$$x[n] = C\alpha^n$$

Note here the cases where  $-1 < \alpha < 0$  (decreasing cone) and  $\alpha < -1$  (increasing cone) behave differently :





The **complex exponential**

**Continuous Complex Exponential**

$$x(t) = Ce^{at} \quad (5)$$

where  $C$  and  $a$  are complex numbers .

An important property is that this signal is **periodic** . This is easier to see if we put it in form (where  $a = 0 + j\omega_0$ ):

$$x(t) = Ae^{j\omega_0 t}$$

with oscillation frequency  $\omega_0$ , and period

$$T = \frac{2\pi}{\omega_0}$$

The discrete version

**Discrete Complex Exponential**

$$x[n] = Ce^{j\Omega n}$$

(sometimes we use  $\Omega$  instead of  $\omega$  for discrete signals)

is only periodic if  $\frac{2\pi}{\Omega}$  is a rational number (see Section 2.3.1). Thus (recall that)  $x[n]$  is periodic iff there exists  $N \in \mathbb{Z}$  (an *integer*  $N$ ) such that  $\Omega N = 2\pi$ . Therefore the periodicity condition is

$$N = \frac{2\pi}{\Omega} = \frac{a}{b}$$

where  $a, b \in \mathbb{Z}$ .

See, for example, Fig. 2 .

The closely related **sinusoidal signal**:

**Sinusoidal signal**

$$x(t) = A \cos(\omega_0 t + \phi) \quad (6)$$

where

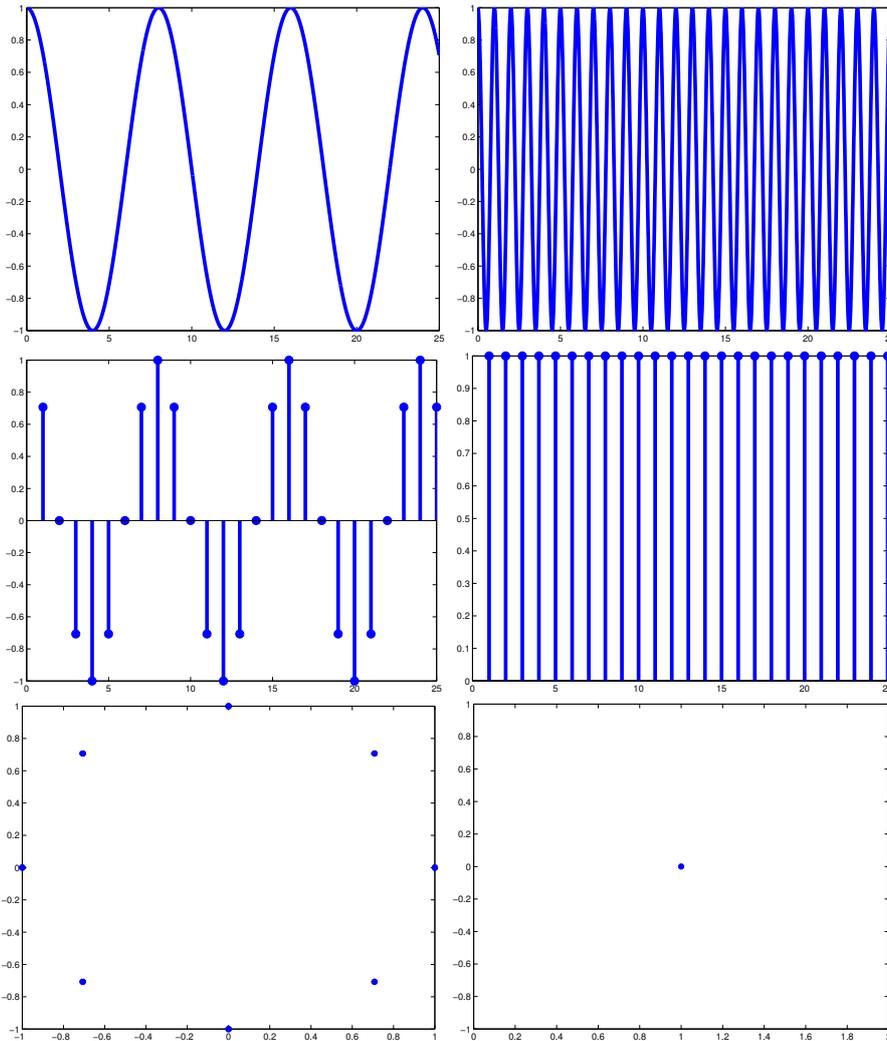


Figure 2: Top: plots with  $\omega = \frac{\pi}{4}$  and  $\omega = 2\pi$  (real part); Middle: the same under DT; Bottom: showing the complex coordinates for the DT waves, plotted on a complex plane – careful with the axes on the bottom-right figure.

- $A$  is **amplitude**
- $t$  time (seconds)
- $\phi$  phase shift (radians)
- $\omega_0$  **frequency** (radians per second )
- $T = \frac{2\pi}{\omega_0}$  is the **fundamental period**, between two peaks

and we often also see  $\omega_0 = 2\pi f_0$  where  $f_0$  is Hertz (cycles per second). Note that the standard sine  $\sin(t)$  and cosine  $\cos(t)$  waves are really just special cases of Eq. 6, where, e.g.  $\omega_0 = 1, \phi = 0$  in  $\cos(t)$  and  $\sin(t) = \cos(t + \phi)$  for  $\phi = -\frac{\pi}{2}$ .

**Relation between complex exponential and sinusoidal signals (Euler's relation).**

$$\begin{aligned}
 x(t) &= Ce^{at} \\
 &= |C|e^{j\phi}e^{at} \\
 &= |C|e^{j\phi}e^{j\omega_0 t} \\
 &= |C|e^{j(\omega_0 t + \phi)} \\
 &= |C|[\cos(\omega_0 t + \phi) + \sin(\omega_0 t + \phi)j] \\
 &= A[\cos(\omega_0 t + \phi) + \sin(\omega_0 t + \phi)j] \\
 &= A\cos(\omega_0 t + \phi) + A\sin(\omega_0 t + \phi)j
 \end{aligned}$$

which is of course periodic! We know (from earlier) that

$$x(t) = e^{j\omega_0(t+T)}$$

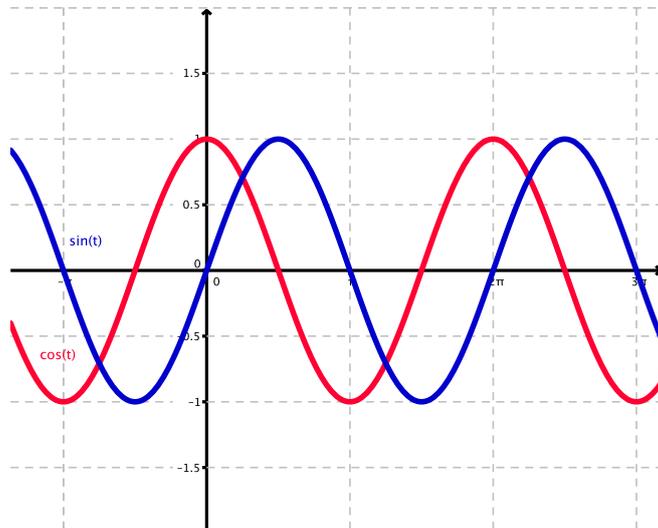
for **period**  $T = \frac{2\pi}{\omega_0}$ .

If, however,  $a = r + j\omega$  is not *purely* imaginary, i.e.,  $r \neq 0$ , we get:

$$\begin{aligned}
 x(t) &= Ce^{at} \\
 &= |C|e^{j\phi}e^{at} \\
 &= |C|e^{j\phi}e^{(r+j\omega_0)t} \\
 &= |C|e^{j\phi}e^{rt}e^{j\omega_0 t} \\
 &= |C|e^{rt}e^{j(\omega_0 t + \phi)} \\
 &= Ae^{rt}\cos(\omega_0 t + \phi) + Ae^{rt}\sin(\omega_0 t + \phi)j
 \end{aligned}$$

which is *not* periodic! It has exponential growth / decay!

Thus, the complex exponential signal is composed of a **real** part and **imaginary** part, which looks like:

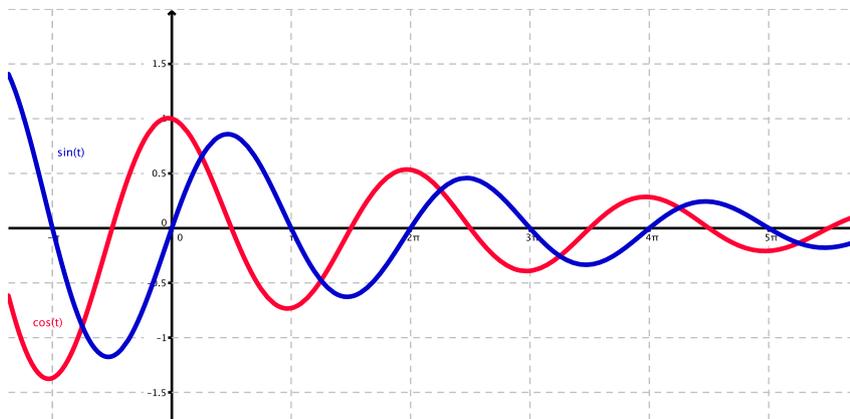


Note that the frequency is the same for all parts, real and imaginary. See also Fig. 8. Now, if we assume then that  $a = r + j\omega_0$  is *not* purely imaginary (i.e., is a complex number with a real part  $r \neq 0$ ), i.e.,

$$x(t) = e^{rt} e^{j\omega_0 t}$$

where  $r > 0$  gives growth and  $r < 0$  gives damping.

For example, for  $\omega_0 = 1, r = -0.1$ , we get:



which is clearly aperiodic, and achieves a **dampening** effect. See also Fig. 8.

These signals can be used to describe the characteristics of a wide variety of physical processes, and they will become very important in Fourier analysis and transforms, etc.

## 2.8 Signals as Vectors

Signals can easily be represented as vectors.

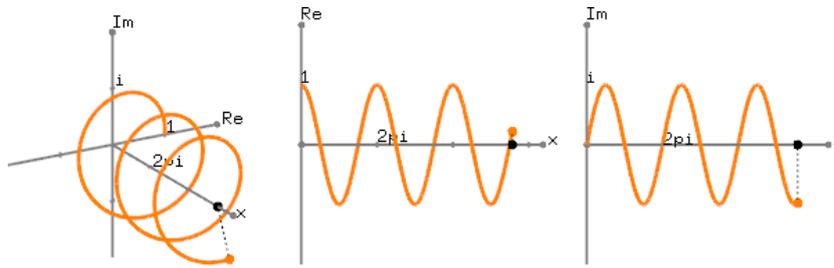


Figure 3: Euler's formula represented graphically. Source: Wikipedia.

Example: The signal  $x[n]$  for  $n = 0, \dots, N - 1$  for  $N = 4$

can be represented as a vector

$$\mathbf{x} = \begin{bmatrix} 1 \\ 4 \\ -2 \\ 0 \end{bmatrix}$$

where  $\mathbf{x} \in \mathbb{R}^N$ .

Even complex signals can be represented as vectors, as

$$\mathbf{x} = \mathbf{x}_{\text{Re}} + j\mathbf{x}_{\text{Im}}$$

where  $\mathbf{x} \in \mathbb{C}^N$ .

We can then carry out many linear algebra operations and **transforms**

- scaling  $\mathbf{y} = \alpha\mathbf{x}$
- addition  $\mathbf{y} = \mathbf{x}_1 + \mathbf{x}_2$
- transpose  $\mathbf{x}^\top$
- etc.

As other vectors, signal vectors are

- **commutative**  $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
- **associative**  $[\mathbf{x} + \mathbf{z}] + \mathbf{y} = \mathbf{x} + [\mathbf{z} + \mathbf{y}]$

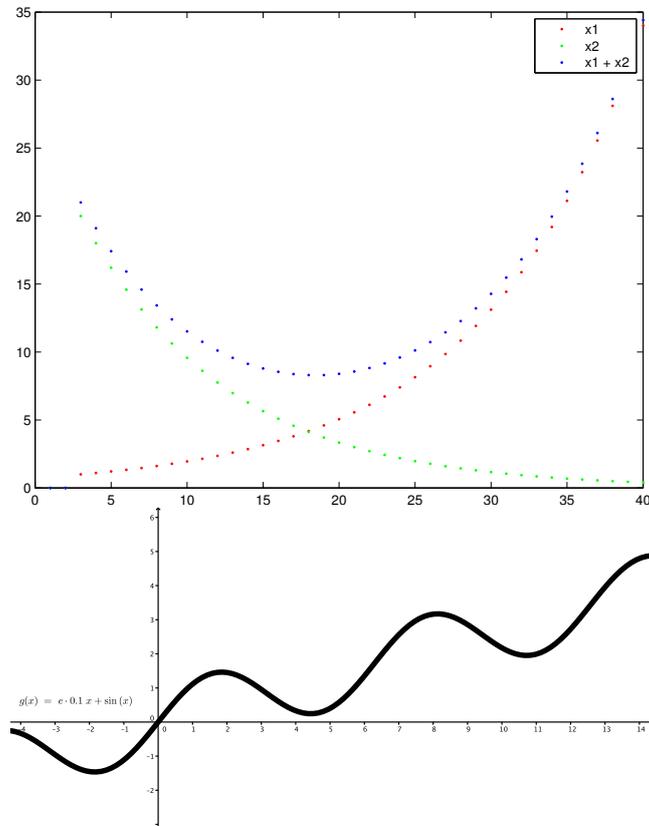


Figure 4: A mix of signals added together. Top:  $x[t] = x_1[n] + x_2[n]$ . Bottom:  $x(t) = e^{0.1t} + \sin(t)$

- **distributive**  $\alpha[\mathbf{x} + \mathbf{y}] = \alpha\mathbf{x} + \alpha\mathbf{y}$

Note the **hermitian transpose** for complex vectors:

$$\mathbf{x} = \begin{bmatrix} 1 + 2i \\ 2 + 3i \\ 3 + 4i \end{bmatrix}, \mathbf{x}^H = [1 - 2i, 2 - 3i, 3 - 4i]$$

We can construct complicated signals by summing unit sample signals that are appropriately *scaled* and *shifted*.

### 3 Systems

### 3.1 Introduction

A **system** is

- Something that transforms signals (hardware or software)
- Representations of physical entities that react to input excitations
- A mathematical **function**: given  $x$ , calculate  $y$ 
  - **independent variable** = time
  - **dependent variable** = voltage, flow rate, sound pressure, etc.

Systems can work in continuous time,

$$x(t) \longrightarrow \boxed{T\{\cdot\}} \longrightarrow y(t)$$

or discrete time,

$$x[n] \longrightarrow \boxed{T\{\cdot\}} \longrightarrow y[t]$$

$T$  is the system. We can do, e.g.,

$$y(t) = T\{x(t)\}$$

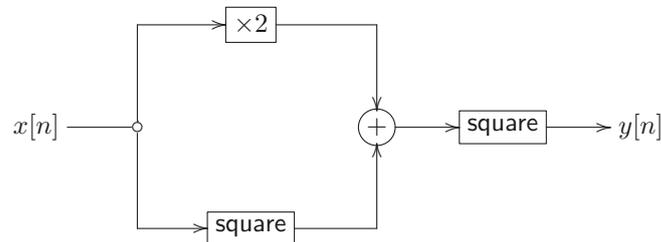
Some examples of systems: :

- input  $\rightarrow$   $\boxed{\text{system}}$   $\rightarrow$  output
- sound in  $\rightarrow$   $\boxed{\text{mobile phone system}}$   $\rightarrow$  sound out
- water in  $\rightarrow$   $\boxed{\text{leaky tank system}}$   $\rightarrow$  water out
- steering wheel angle  $\rightarrow$   $\boxed{\text{car}}$   $\rightarrow$  position in lane
- position of hand  $\rightarrow$   $\boxed{\text{mass and spring system}}$   $\rightarrow$  position of mass
- spoken English  $\rightarrow$   $\boxed{\text{interpreting system}}$   $\rightarrow$  spoken Spanish
- etc.

Systems can be modular :

- sound in  $\rightarrow$   $\boxed{\text{mobile A}}$   $\rightarrow$   $\boxed{\text{tower 1}}$   $\rightarrow$   $\boxed{\text{tower 2}}$   $\rightarrow$   $\boxed{\text{mobile B}}$   $\rightarrow$  sound out
- spoken English  $\rightarrow$   $\boxed{\text{transcription}}$   $\rightarrow$   $\boxed{\text{translation}}$   $\rightarrow$   $\boxed{\text{synthesizing}}$   $\rightarrow$  spoken Spanish

Example: Consider the **second-order system**



It can be represented by

$$y[n] = (2 \cdot x[n] + x[n]^2)^2$$

So for  $x[n] = 1$ ,

$$y[n] = (2 \cdot 1 + 1^2)^2 = 9$$

and for  $x[n] = 3$ ,

$$y[n] = 225$$

and so on for any  $n \in \mathbb{Z}$ .

**Sampling and Reconstruction** **Sampling**: converting CT signals to DT signals, e.g.,

- audio  $\rightarrow$  encode  $\rightarrow$  mp3
- image  $\rightarrow$  encode  $\rightarrow$  jpg

and **reconstruction** is the opposite (converting DT to CT signals).

- zero-order hold method
- piecewise linear

### Ways to represent systems

1. verbal description, e.g.,

“the output is the input plus the previous input”

2. **difference equation** (for DT systems)

$$y[n] = x[n] + x[n - 1]$$

or **differential equation** (for CT systems)

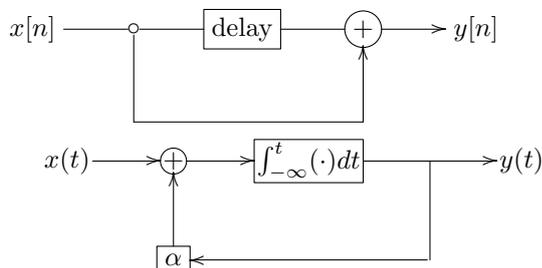
$$\frac{dy(t)}{dt} = x(t) + \alpha y(t)$$

either *imperative*, as above, telling what the system does, or *declarative*,

$$x[n] = y[n] - x[n - 1]$$

telling the rules of the system.

3. **block diagram:**



- signals = primitives
- operators = means of combination  
can build many systems from three basic operators:

- $\oplus$  adder;  $X_1 + X_2$
- $\alpha$  scale/gain<sup>1</sup>;  $c \cdot X$
- $\text{delay}$  delay;  $\mathcal{R}X$

$x[n]$	$x[n-1]$	$x[n-2]$
$X$	$\mathcal{R}X$	$\mathcal{R}^2X$
$[0, 1, 0, 0, 0]$	$[0, 0, 1, 0, 0]$	$[0, 0, 0, 1, 0]$

In CT systems, we use the **integrator** operator  $\int_{-\infty}^t (\cdot) dt$  instead of  $\text{delay}$ .

4. operator representation (systems as polynomials), e.g.,

$$Y = (1 + \mathcal{R})X \\ = X + \mathcal{R}X$$

(with right shift / 'delay' operator  $\mathcal{R}$ .) Note the equivalence to the differential equation above.

5. tabular outputs:

$n$	$x[n]$	$x[n-1]$	$y[n]$
-1	0	0	0
0	1	0	1
1	1	1	2
2	0	1	1
3	0	0	0

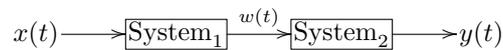
<sup>1</sup>Normally drawn in a triangle, rather than in a square

## 3.2 Interconnection of Systems

- Series (cascade)
- Parallel
- Feedback

Combinations of systems can be seen as a single system.

### 3.2.1 Series (cascade)



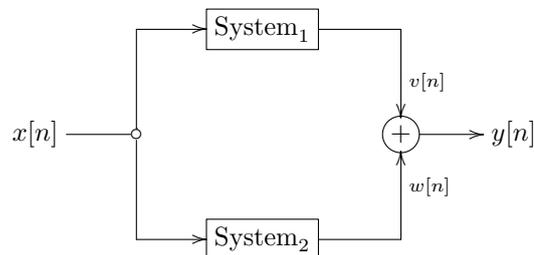
If System 1 multiplies by 3 and System 2 multiplies by 4 then

$$y(t) = 4w(t) = 4(3(x(t))) = 12x(t)$$

for example, with  $x[n] = 2$ ,

$$\begin{aligned}w(t) &= 3(x(t)) \\y(t) &= 4(w(t)) \\ &= 12(x(t))\end{aligned}$$

### 3.2.2 Parallel



If System 1 multiplies by 3 and System 2 multiplies by 4 then

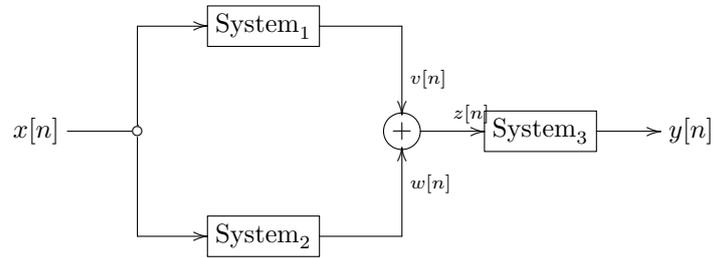
$$\begin{aligned}y[n] &= w[n] + v[n] \\ &= 4x[n] + 3x[n] \\ &= x[n][3 + 4] = 7x[n]\end{aligned}$$

for example, with  $x[n] = 2$ ,

$$\begin{aligned}w[n] &= 3(x[n]) = 3 \cdot 2 = 6 \\v[n] &= 4(x[n]) = 4 \cdot 2 = 8 \\y[n] &= w[n] + v[n] = 6 + 8 = 14 \\ &= 3(x[n]) + 4(x[n]) = 6 + 8 = 14 \\ &= x[n](3 + 4) \\ &= 7x[n] = 7 \cdot 2 = 14\end{aligned}$$

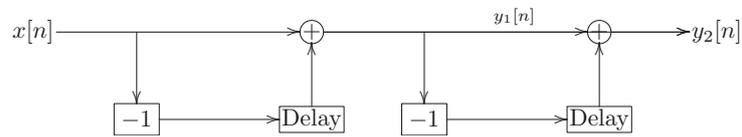
We could now make a new, single, system with just one operator:  $\times 7$ .

### 3.2.3 Combination



We can represent block diagrams with algebraic expressions and vice versa. Thus we can modify the algebra to create a new diagram representation, and vice versa.

**Example:** Consider the system:



representable in **operator notation** as:

$$Y_1 = (1 - \mathcal{R})X$$

$$Y_2 = (1 - \mathcal{R})Y_1$$

Note that we can substitute  $Y_1$  into  $Y_2$ :

$$\begin{aligned} Y_2 &= (1 - \mathcal{R})(1 - \mathcal{R})X \\ &= (1 - \mathcal{R})^2 X \\ &= (1 - 2\mathcal{R} + \mathcal{R}^2)X \end{aligned}$$

We can do the same thing with **difference equations**

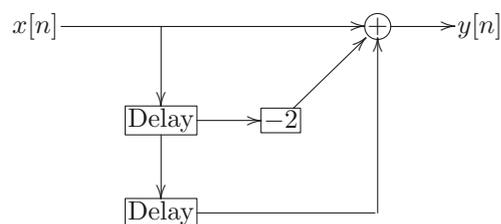
$$y_1[n] = x[n] - x[n - 1]$$

$$y_2[n] = y_1[n] - y_1[n - 1]$$

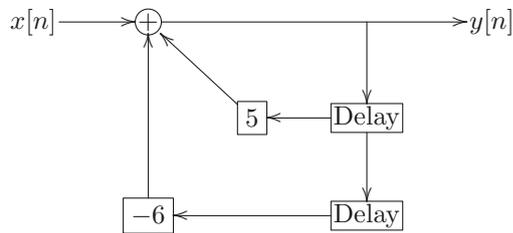
substituting the first into the second to get:

$$\begin{aligned} y_2[n] &= x[n] - x[n - 1] - (x[n - 1] - x[n - 2]) \\ &= x[n] - x[n - 1] - x[n - 1] + x[n - 2] \\ &= x[n] - 2x[n - 1] + x[n - 2] \end{aligned}$$

Now, we can rebuild the system under a new representation:



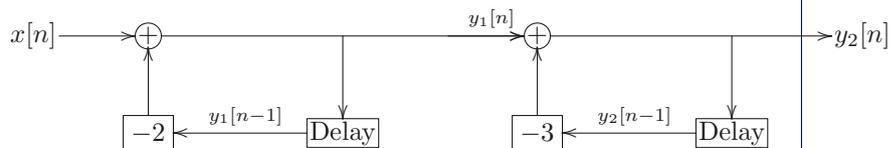
**Example:** Consider the system



Under operator notation, we can derive

$$\begin{aligned}
 Y &= X + 5\mathcal{R}Y - 6\mathcal{R}^2Y \\
 X &= Y - 5\mathcal{R}Y + 6\mathcal{R}^2Y \\
 &= Y(1 - 5\mathcal{R} + 6\mathcal{R}^2) \\
 &= Y(1 - 2\mathcal{R})(1 - 3\mathcal{R})
 \end{aligned}$$

And, thus, we could draw this also as a cascade system:.



### 3.2.4 Feedback

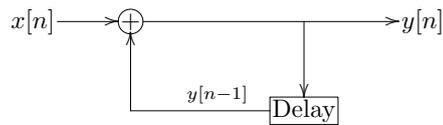
We just saw an example of a feedback system. Another important example (which should look familiar) is

**The accumulator**

$$y[n] = \sum_{k=-\infty}^n x[k] \quad (7)$$

which will convert  $x[n] = \delta[n]$  into  $y[n] = u[n]$ .

$$x[n] \rightarrow \left[ \sum_{k=-\infty}^n x[k] \right] \rightarrow y[n]$$



$$y[n] = x[n] + y[n - 1]$$

$n$	$x[n]$	$y[n - 1]$	$y[n]$
0	1	0	1
1	2	1	3
2	2	3	5
3	0	5	5
4	0	5	5

Note that we typically assume that systems *start at rest*.

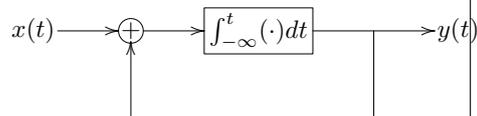
The CT version of the accumulator is

**The integrator**

$$y(t) = \int_{-\infty}^t x(\tau) d\tau \quad (8)$$

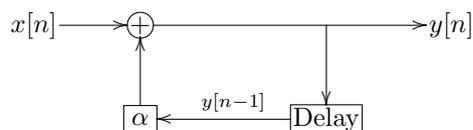
$$x(t) \rightarrow \left[ \int_{-\infty}^t x(\tau) d\tau \right] \rightarrow y(t)$$

which will convert  $x(t) = \delta(t)$  into  $y(t) = u(t)$ .



Note that in CT systems we use integration instead of delays.

Adding **gain**, we get a feedback system like



where the behaviour of the system depends on the value of  $\alpha$  (growth if  $\alpha > 1$ , decay if  $0 < \alpha < 1$ , remain the same if  $\alpha = 1$ ). . We can write out this system as

$$\underbrace{y[n]}_{\text{output}} = \underbrace{\alpha y[n-1]}_{\text{growth/decay}} + \underbrace{x[n]}_{\text{addition}}$$

Feedback systems are important in, for example, automatic control systems like temperature-control or auto-piloting of vehicles / aircraft. .

**Example:** The above feedback system can represent a bank balance. We can write it as

$$\underbrace{y[n]}_{\text{balance}} = \underbrace{\alpha y[n-1]}_{\text{prev w/ interest}} + \underbrace{x[n]}_{\text{net deposit}}$$

where  $\alpha = 1 + r$  for interest rate  $r$ , e.g.,  $r = 0.005$ . Equivalently,

$$y[n] - \alpha y[n-1] = x[n]$$

i.e., the net deposit since  $n - 1$  is the same as the current balance at  $n$  minus the balance at time  $n - 1$  times the interest rate.

For working with feedback systems, we will need to review **geometric series** and **Taylor series expansion**.

#### Deriving a closed form equation for a feedback system

$$\begin{aligned} y_n &= \alpha y_{n-1} \\ &= \alpha(\alpha y_{n-2}) \\ &= \alpha(\alpha(\alpha y_{n-3})) \\ &= \dots \\ &= \alpha(\alpha(\alpha \dots (\alpha y_0))) \\ &= \alpha^n y_0 \\ &= y_0 \alpha^n \end{aligned}$$

which is an exponential signal! Note that  $y_n$  is just an alternative notation for  $y[n]$ .

**Example:** A first order difference equation; population growth. A town with initial population  $p_0 = 300$  and growth rate  $r = 0.01$ . In 60 years, what will the population be? The population at  $p_n$  is

$$\begin{aligned} p_n &= (1 + r)p_{n-1} \\ &= \alpha p_{n-1} \\ &= p_0 \alpha^n \\ &= p_0(1 + r)^n \end{aligned}$$

Therefore the population at  $p_{60}$  will be

$$p_0(1 + r)^n = 300(1 + 0.01)^{60} = 602$$

$n$	$p[n - 1]$	$p[n] = (1 + r)p[n - 1] = y[0](1 + r)^n$
1	300.00	303.00
2	303.00	306.03
3	306.03	309.09
3	306.03	309.09
...	...	...
70	596.07	602.03

Thus we see the population growing exponentially. However, this model does not take into account immigration or emigration (e.g.,  $+x_n$  and  $-x_n$ , respectively).

**Example:** A mortgage, at fixed 0.0274 (annual) interest rate ( $r = \frac{0.0274}{12}$  monthly), will grow from  $y[0] = \$67,200$  (initial debt), as:

$$y[n] = (1 + r)y[n - 1]$$

for  $n$  months (we already know this from the earlier examples) and would grow to  $y[30 \cdot 12] = 152,740$  after  $N = 30 \cdot 12$  months (30 years). Making regular payments of  $x[n] = 300$ , it will grow as

$$y[n] = (1 + r)y[n - 1] - x[n]$$

(from here on we assume constant  $x = x[n] = 300$  repayments). After 30 years we get

$$y[N] = -14,172$$

Oops, too many payments.

**Deriving a closed form equation for a feedback system with linear emigration**

Assuming some constant  $x$  and  $\alpha$ , we have

$$y_1 = \alpha y_0 - x$$

$$y_2 = \alpha y_1 - x$$

$$= \alpha(\alpha y_0 - x) - x$$

$$y_3 = \alpha(\alpha(\alpha y_0 - x) - x) - x$$

$$\dots = \dots$$

$$y_N = \alpha^N y_0 - (\alpha^{(N-1)}x - \dots - \alpha x - x)$$

$$= y_0 \alpha^N - x \sum_{i=0}^{N-1} \alpha^i$$

$$= y_0 \alpha^N - x \frac{\alpha^N - 1}{\alpha - 1}$$

**Example:**

To pay off the mortgage from the previous example in 5 years, what monthly payments should be made?

First we use the formula from above.

$$\begin{aligned} y_N &= y_0(1+r)^N - x \frac{(1+r)^N - 1}{(1+r) - 1} \\ &= y_0(1+r)^N - x \frac{(1+r)^N - 1}{r} \end{aligned}$$

The loan will be paid back when  $y[N] = 0$ , which implies

$$0 = y_0(1+r)^N - x \frac{(1+r)^N - 1}{r}$$

Solving for  $x$  gives us:

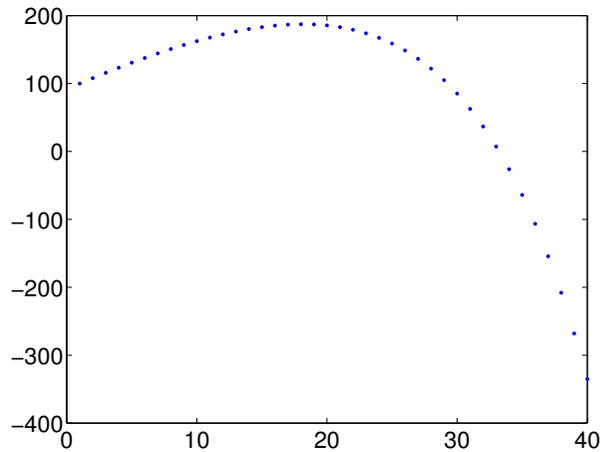
$$x = y_0 \frac{r(1+r)^N}{(1+r)^N - 1}$$

To pay back in 5 years, we plug in  $N = 5 \cdot 12 = 60$  months, and get

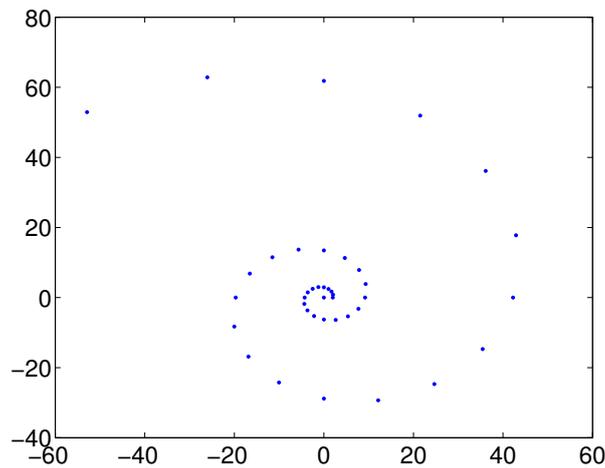
$$x = 1,199.7$$

Payments of \$1,199.70 / month will pay off the mortgage in 5 years.

We can do more complicated things, e.g.,



where  $x[n] = (1.001t)$ ; and  $y[n] = \alpha y[n - 1] - x[n]$ . When  $\alpha$  is complex, we get a crazy spiral:



$$a = 1.1e^{j\pi/8} \approx 1.0163 + 0.4210j$$

which is looking at the complex plain with an exponentially increasing amplitude.

### 3.3 Properties

#### 3.3.1 Memory / Memoryless

A system has **memory** if its output at time  $t_0, y(t_0)$ , depends on input values *other than*  $x(t_0)$ . Conversely, a system is **memoryless** if  $y[n]$  depends *only* on  $x[n]$ .

The **integrator** has memory:

$$y(t) = \int_{-\infty}^t x(\tau) d\tau$$

as does a system

$$y(t) = x(t + 5)$$

The gain system is a *memoryless* system (aka *static* system),

$$y(t) = Kx(t)$$

as well as the “squarer”:

$$y(t) = x^2(t)$$

### 3.3.2 Invertibility

An **invertible system** is a deterministic system where distinct inputs result in distinct outputs. The “squarer”

$$y(t) = x(t)^2$$

is *not* invertible, because, e.g., for  $y(t) = 4$ , there are two possible inputs: 2 and  $-2$ . The gain system

$$y(t) = Kx(t)$$

is invertible, because

$$x(t) = \frac{1}{K}y(t)$$

and has a unique  $x(t)$  for each  $y(t)$ .

We can define the **identity system** to be a system for which the input equals the output. The **inverse** of a system  $T$  is a second system  $T_i$  that, when cascaded with  $T$ , yields the identity system:

$$x(t) \rightarrow \boxed{T} \rightarrow \boxed{T_i} \rightarrow x(t)$$

the identity system for the gain example above is

$$x(t) \rightarrow \boxed{K(\cdot)} \rightarrow \boxed{\frac{1}{K}(\cdot)} \rightarrow x(t)$$

### 3.3.3 Causality

A causal system does *not* anticipate future actions (i.e., the value at  $n_0$  depends only on the input sequence for  $n \leq n_0$ ). For example

$$y(t) = x(t + 1)$$

is *not* causal, whereas

$$y(t) = x(t - 1)$$

is causal, because it does *not* take into account future inputs (at  $> t$ ) to produce an output at time  $t$ .

All physical systems are causal, but in [non-real-time / delayed] filtering, for example, we can take into account future events, since they have already been recorded. We can have non-causal components in a causal system:

$$x(t) \rightarrow \boxed{\text{delay 30s}} \rightarrow \boxed{\text{advance 25s}} \rightarrow x(t - 5)$$

### 3.3.4 Stability

A system is stable when small divergences in the input don't matter in the output. If a stable system is used "responsibly" (i.e., with *bounded* input) it will not blow up. Stable systems are difficult to upset.

We can say a system is **BIBO stable** if a Bounded Input always produces a Bounded Output.

#### Testing a system is [BIBO] stability

If we can bound a signal  $x(t)$  to say that for a *positive finite* value  $B_x$

$$|x(t)| < B_x$$

for all time  $t$ , produces a *fixed positive finite* value  $B_y$

$$|y(t)| < B_y$$

for all time  $t$ . In other words, for *all* bounded inputs there is a bounded output.

To say that a system is BIBO unstable, we need to find just one bounded input for which there is an unbounded output.

Important: for the system to have a property, it must have the property for *all* inputs. However, note that only the amplitude must be finite (not time); and must be finite for all time.

🔍 **Example:** Testing the **gain** system for stability:

$$y(t) = Gx(t)$$

We pick some positive finite input  $B_x$ , which produces a bounded output

$$|G \cdot B_x| < B_y$$

where, e.g.,  $B_y = G \cdot B_x + 1$ , for *any* bounded  $B_x, B_y$ . This system is stable.

☞ **Example:** Testing the accumulator (see Eq. 7) for stability. Consider inputting  $x[n] = u[n]$ , clearly bounded by  $B_x = 1$  over all time. The output is:

$$y[n] = \sum_{k=-\infty}^n u[k]$$

There is no finite choice for  $B_y$  such that  $B_y < \infty$  (positive finite number) *for all time*  $n$ . Recall, to show that a system is *not* stable, we just need to find a *single* exception. Thus the accumulator is an **unstable** system.

The same applies to the CT accumulator:

$$y(t) = \int_{-\infty}^t u(t) dt = \int_0^t 1 dt = t$$

(over all time  $t$ ). So for bound  $B_x$ , we cannot find a bounded  $B_y$  *over all time*; the system is **unstable**.

An unstable system is not of much value, since it can be difficult to control!

### 3.3.5 Time invariance

If input goes into the system tomorrow instead of today, the result will be the same. Formally, in a **time invariant** system (aka, **fixed** system),

$$y(t - t_0) = T[x(t - t_0)]$$

#### Testing a system for time [in]variance

First:

1. Input  $x(t)$  to the system and obtain  $y(t)$ , then shift  $y(t)$  in time (replace all  $t$  with  $t - t_d$ ) – this is  $y(t - t_d)$
2. Shift  $x(t)$  in time (replace all  $t$  with  $t - t_d$ ), input this  $x(t - t_d)$  into the system and obtain  $y_d(t)$  out.

Now compare  $y(t - t_d)$  and  $y_d(t)$  – are they the same? If they are, then the system is time invariant.

☞ **Example:** Is the following system time invariant?

$$y(t) = e^{x(t)}$$

1. run  $x(t)$  through the system, we get  $y(t) = e^{x(t)}$ ; next add a delay, to get  $y(t - t_d) = e^{x(t - t_d)}$ .
2. run  $x(t - t_d)$  through the system, we get  $y_d(t) = e^{x(t - t_d)}$ ; compare with above.

Yes, the system is time invariant.

☞ **Example:** Is the following system time invariant?

$$y(t) = \cos(\omega_0 t)x(t)$$

1. run  $x(t)$  through the system, we get  $y(t) = \cos(\omega_0 t)x(t)$ ; next add a delay, to get  $y(t - t_d) = \cos(\omega_0(t - t_d))x(t - t_d)$
2. run  $x(t - t_d)$  through the system, we get  $y_d(t) = \cos(\omega_0 t)x(t - t_d)$ ; compare with above.

$$y(t - t_d) \neq y_d(t)$$

No, the system is *not* time varying.

☞ **Example:**

Is the accumulator (see Eq. 7) time invariant?

1. Run  $x[n]$  through the system; add a delay to it (replace  $n$  with  $n - n_d$  in  $y[n]$ ):

$$y[n - n_d] = \sum_{k=-\infty}^{n-n_d} x[k]$$

2. Run  $x[k - n_d]$  through the system (input  $x[k - n_d]$  instead of  $x[k]$ ):

$$\begin{aligned} y_d[n] &= \sum_{k=-\infty}^n x[k - n_d] \\ &= \sum_{k_1=-\infty}^{n-n_d} x[k_1] \text{ (replace } k \text{ with } k_1 = k - n_d) \end{aligned}$$

Now it is equal to the first equation (recall that  $k$  and  $k_1$  are just dummy variables)! Therefore this system *is* time invariant.

### 3.3.6 Linearity

This is the most important of the properties that we look at. A **linear system** has following properties:

- **Additivity:** for inputs  $x_1(t)$  and  $x_2(t)$ :

$$x_1(t) + x_2(t) \rightarrow \boxed{\text{S}} \rightarrow y_1(t) + y_2(t)$$

- **Homogeneity:** for some constant  $a$ ,

$$ax(t) \rightarrow \boxed{\text{S}} \rightarrow ay(t)$$

for some constant  $a$ .

Thus, with constants  $a_k, k = 1, 2$ , we have

$$a_1x_1(t) + a_2x_2(t) \rightarrow \boxed{\text{S}} \rightarrow a_1y_1(t) + a_2y_2(t)$$

No physical system is linear under all operating conditions, but under some conditions it is.

An amplifier is a linear system:

$$y(t) = Kx(t)$$

but

$$y(t) = x^2(t)$$

is not linear.

**Example:** Is the following a linear system?

$$x(t) \rightarrow \boxed{\text{squarer}} \rightarrow y(t) = [x(t)]^2$$

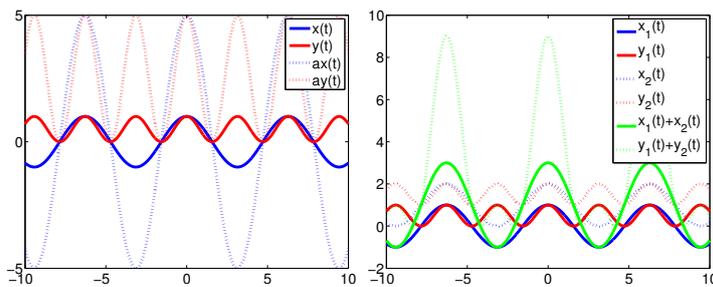
Testing for **homogeneity**:

$$\begin{aligned} ax(t) &\rightarrow ay(t) \\ \underbrace{ax(t)}_{\text{input}} &\rightarrow [ax(t)]^2 = a^2 \underbrace{x^2(t)}_{y(t)} = a^2 y(t) \end{aligned}$$

And since  $ay(t) \neq a^2 y(t)$ , it's not homogeneous! Already *not* linear, but next step would be testing for **additivity**:

$$\begin{aligned} x_1(t) + x_2(t) &\rightarrow y_1(t) + y_2(t) \\ \rightarrow \underbrace{[x_1(t) + x_2(t)]^2}_{\text{input}} &= \underbrace{x_1^2(t)}_{y_1(t)} + 2x_1(t)x_2(t) + \underbrace{x_2^2(t)}_{y_2(t)} \end{aligned}$$

but since  $y_1(t) + y_2(t) \neq y_1(t) + 2x_1(t)x_2(t) + y_2(t)$ , it's *not* additive! This makes sense; think of  $x(t) = \cos(t)$  (and  $y(t) = \cos^2(t)$ ):



Example: Testing the properties of a particular system

$$y(t) = \sin(2t)x(t)$$

This system is

1. **memoryless**, because the output is a function of the input at the present time;
2. **not invertible**, because at  $t = \pi$ , I can put in any input I want but *not* get distinct output (it's always 0);
3. **causal**, because the output does not depend on any future input.
4. **stable**, because the output is bounded for all bounded inputs ( $\sin(2t)$  is periodic, no matter what bounded input I put in, I will always have bounded output –  $< 1$  in this case);
5. **time varying**, because from we see that

$$y_d(t) = \sin(2t)x(t - t_0) \neq y(t - t_0) = \sin(2(t - t_0))x(t - t_0)$$

6. and **linear**, since

$$\begin{aligned} a_1x_1(t) + a_2x_2(t) &\rightarrow \sin(2t)[a_1x_1(t) + a_2x_2(t)] \\ &= a_1\sin(2t)x_1(t) + a_2\sin(2t)x_2(t) \\ &= a_1y_1(t) + a_2y_2(t) \end{aligned}$$

### 3.4 Linear Time-Invariant Systems (LTIS)

A Linear Time-Invariant System (LTIS) is a system which is

- **Linear** (properties of **homogeneity** and **additivity**); and
- **Time invariant**

Why are LTI systems interesting to us?

- Able to model so many real-world systems
- ... and can even approximate many non-linear systems with LTI systems
- We can do transforms on them: **Laplace transform, Fourier transform.**

#### 3.4.1 Linearity

A linear system has the properties of

- **Additivity**

$$x_1(t) + x_2(t) \rightarrow \boxed{\text{S}} \rightarrow y_1(t) + y_2(t)$$

- **Homogeneity**: for some constant  $a$ ,

$$ax(t) \rightarrow \boxed{\text{S}} \rightarrow ay(t)$$

which together, define

- **Linearity**: for constants  $a_k, k = 1, 2$

$$a_1x_1(t) + a_2x_2(t) \rightarrow \boxed{\text{S}} \rightarrow a_1y_1(t) + a_2y_2(t)$$

Therefore, if a system has these properties, we can say that it's linear. If we know the output for a signal  $x(t)$ , we also know the output for a signal composed of a linear combination of shifted versions of  $x(t)$ . If we scale the inputs  $x_1(t), x_2(t), \dots$  we scale the outputs, i.e., if the input is

$$x(t) = a_1x_1(t) + a_2x_2(t) + a_3x_3(t) + \dots$$

the output is given by

$$y(t) = a_1y_1(t) + a_2y_2(t) + a_3y_3(t) + \dots$$

in other words, letting  $y(t) = H[x(t)]$ , for a linear system,

$$y(t) = H[a_1x_1(t) + a_2x_2(t) + a_3x_3(t)] = a_1y_1(t) + a_2y_2(t) + a_3y_3(t)$$

where  $H$  is the **transfer function** variable

$$y(t) = H[x(t)]$$

which indicates how inputs *transfer* to outputs.

### 3.4.2 Impulse response

$h(t)$  is the system **impulse response**; the system's response to the impulse  $\delta(t)$ ; it *completely characterizes* the LTIS:

$$\text{impulse } \delta(t) \longrightarrow \boxed{\text{LTIS } h(t)} \longrightarrow \text{impulse response } h(t)$$

We can think of the LTIS as a **black box** completely characterized by  $h(t)$ ; it is the system output due to an impulse sequence  $\delta(t)$  at the system input. The same goes for DT; given  $\delta[n]$  input into a system, the system responds with  $h[n]$ :

$$\underbrace{x[n] = \delta[n]}_{\text{impulse}} \rightarrow \boxed{\text{LTIS}} \rightarrow \underbrace{y[n] = h[n]}_{\text{impulse response}} = \underbrace{H\{\delta[n]\}}_{\text{transfer function}}$$

$$\underbrace{\delta[n]}_{\text{impulse}} \rightarrow \boxed{\text{LTIS}} \rightarrow \underbrace{h[n]}_{\text{impulse response}} = \underbrace{H\{\delta[n]\}}_{\text{transfer function}}$$

If we multiply the impulse sequence by a constant  $\alpha$  then, by homogeneity, the output is also multiplies by  $\alpha$ , i.e.,

$$\alpha\delta[n] \rightarrow \boxed{\text{LTIS}} \rightarrow \alpha h[n]$$

and, by time invariance,

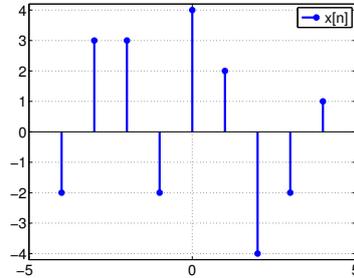
$$\alpha\delta[n - k] \rightarrow \boxed{\text{LTIS}} \rightarrow \alpha h[n - k]$$

for any delay  $k = -\infty, \dots, -3, -2, -1, 0, 1, 2, 3, \dots, \infty$ . So we have:

$$\underbrace{\alpha\delta[n - k]}_{\text{impulse}} \rightarrow \boxed{\text{LTIS}} \rightarrow \underbrace{\alpha h[n - k]}_{\text{impulse response}}$$

Therefore, we can represent any arbitrary input sequence in terms of impulse response sequence as follows (as we saw in Eq. 3):

**Example:** A signal defined as a *sum of shifted and scaled impulses* (from Eq. 3)



$$\begin{aligned}
 x[n] &= \dots + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] \dots \\
 &= \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]
 \end{aligned}$$

i.e., a *sum of shifted and scaled impulses* (where  $x[k]$  is acting like a *scalar*  $a_k$ ). We can write out :

$x[4]\delta[n-4]$	[0 0 0 0 0 0 0 1]
$x[3]\delta[n-3]$	[0 0 0 0 0 0 -2 0]
$x[2]\delta[n-2]$	[0 0 0 0 0 0 -4 0 0]
$x[1]\delta[n-1]$	[0 0 0 0 0 2 0 0 0]
$x[0]\delta[n]$	[0 0 0 0 4 0 0 0 0]
$x[-1]\delta[n+1]$	[0 0 0 -2 0 0 0 0 0]
$x[-2]\delta[n+2]$	[0 0 3 0 0 0 0 0 0]
$x[-3]\delta[n+3]$	[0 3 0 0 0 0 0 0 0]
$x[-4]\delta[n+4]$	[-2 0 0 0 0 0 0 0 0]

where,

$x[-4]$	$x[-4]\delta[n-4]$	$-2 \cdot 1$	$-2$
$x[-3]$	$x[-3]\delta[n-3]$	$3 \cdot 1$	$3$
$x[k]$	$x[k]\delta[n-k]$	$\cdot$	$\cdot$

Thus, any LTI System is *completely defined* by  $h[n]$ , the impulse response:

$$\underbrace{x[n]}_{\text{input}} \rightarrow \underbrace{h[n]}_{\text{black box}} \rightarrow \underbrace{y[n]}_{\text{output}}$$

therefore if we know  $h[n]$ , we can get the output  $y[n]$  from any input  $x[n]$  (via **convolution**)!

### 3.5 Convolution

The output of an LTIS can be calculated as the convolution of the input  $x(\cdot)$  with the **impulse response**  $h(\cdot)$ . The CT case is just a generalization of the

DT case:

$$\underbrace{\delta(t)}_{\text{impulse}} \rightarrow \boxed{\text{LTIS}} \rightarrow \underbrace{h(t)}_{\text{impulse response}}$$

and since we know (as a time-invariant system) that

$$\underbrace{\delta(t-a)}_{\text{impulse}} \rightarrow \boxed{\text{LTIS}} \rightarrow \underbrace{h(t-a)}_{\text{impulse response}}$$

we can write any input as shifted impulses,

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

which brings us to, the

**convolution integral**

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \quad (9)$$

In the DT case, we have the

**convolution sum**

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] \quad (10)$$

which should look familiar from the previous example. The notation  $x(t) * h(t)$  (or  $x[n] * h[n]$ ) is the notation for **convolution**.

☞ **Note:** A more intuitive (but lesser used) notation is:  $(x * h)[n]$  instead of  $x[n] * h[n]$ , since convolution is *not* an operation on samples, but rather, signals (for any time  $n$ )! In all cases be careful not to confuse “\*” with the multiplication symbol!

Basically, we have an input  $x(t)$ , we can use convolution to get the output  $y(t)$  from an LTIS defined by  $h(t)$ :

$$y(t) = x(t) * h(t)$$

and in the discrete case,

$$y[n] = x[n] * h[n]$$

In other words, the output of a system is the convolution of the input and the impulse response. If we have the impulse response (the system’s response to the unit impulse  $\delta[n]$ ), we can get the output for *any* entry signal  $x(t)$ !

### Convolution steps (DT)

To convolute  $x[n] * h[n]$ ; first, write both as functions of  $k$ ,

$$y[n] = \sum_k x[k]h[n - k]$$

then

1. **Flip**  $h[k]$  (to  $h[-k]$ )
2. **Shift**  $h[n - k]$  by changing  $n$  from  $n = -\infty$  until there is overlap (where both are non-zero at some point in time) between  $h[n - k]$  and  $x[k]$  begins
3. **Calculate** the sum of the multiplication of these two vectors

$$y[n] = \sum (h[n - k] \cdot x[k])$$

4. **Step**  $n = n + 1$
5. **Repeat** from Step 3 until overlap ends.

In other words, convolution is like a **sliding weighted average** of  $x[n]$ , weighted by  $h[n]$ .

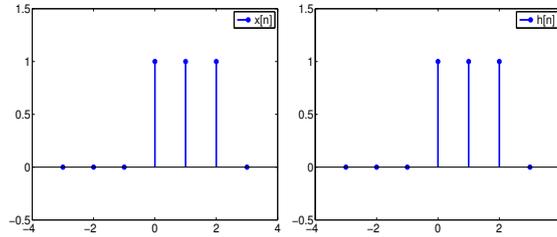
☞ **Note:** Due to the commutative property of convolution

$$h[n] * x[n] \equiv x[n] * h[n]$$

and therefore we could flip and shift  $x[n - k]$ , and multiply with  $h[n]$  (all instances of each reverse) and get the same result.

Notice that, if  $x[n] = \delta[n]$ , the result is  $y[n] = h[n]$ !

Example:  $y[n] = x[n] * h[n]$ , where



First, write/draw as  $x[k]$  and  $h[n-k]$ ;  $\begin{array}{r} x[k] \\ h[k] \end{array} \begin{array}{r} 0 \ 0 \ 1 \ 1 \ 1 \\ 0 \ 0 \ 1 \ 1 \ 1 \end{array}$  then  
for all  $n$  where there is overlap between  $x[k]$  and  $h[n-k]$ :

- for  $n = 0$ :
 

$n$	-2	-1	0	1	2
$h[0-k]$	1	1	1	0	0
$x[k] \times h[0-k]$	0	0	1	0	0

 $y[0] = (\sum x[k] \times h[0-k]) = 1$
- for  $n = 1$ :
 

$n$	-2	-1	0	1	2
$h[1-k]$	0	1	1	1	0
$x[k] \times h[1-k]$	0	0	1	1	0

 $y[1] = (\sum x[k] \times h[1-k]) = 2$
- for  $n = 2$ :
 

$n$	-2	-1	0	1	2
$h[2-k]$	0	0	1	1	1
$x[k] \times h[2-k]$	0	0	1	1	1

 $y[2] = (\sum x[k] \times h[2-k]) = 3$
- for  $n = 3$ :
 

$n$	-2	-1	0	1	2
$h[3-k]$	0	0	0	0	1
$x[k] \times h[3-k]$	0	0	0	1	1

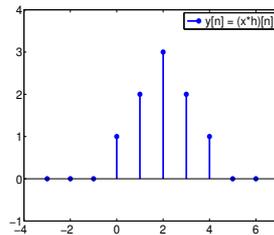
 $y[3] = (\sum x[k] \times h[3-k]) = 2$
- for  $n = 4$ :
 

$n$	-2	-1	0	1	2
$h[4-k]$	0	0	0	0	1
$x[k] \times h[4-k]$	0	0	0	0	1

 $y[4] = (\sum x[k] \times h[4-k]) = 1$

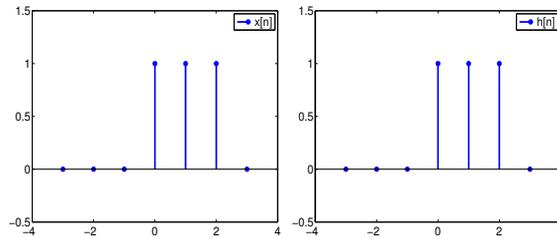
And therefore, we get:

$n$	-2	-1	0	1	2	3	4	5
$y[n]$	0	0	1	2	3	2	1	0



Note that (in DT time) if  $x$  has  $n_x$  samples, and  $h$  has  $n_h$  samples,  $y$  has  $n_y = n_x + n_h - 1$  samples.

**Example:**



1. **Flip**

$$h[-k] = [11100]$$

2. **Shift**  $h[n - k]$  (by changing  $n$ ) to where overlap begins

$$h[0 - k] = [11100]$$

(i.e., overlap begins at  $n = 0$ ).

$$n = 0$$

3. **Calculate**

$$y[0] = \sum (h[0 - k] \cdot x[k]) = 1$$

4. **Step**  $n = n + 1 = 1$

5. **Calculate**

$$y[1] = \sum (h[1 - k] \cdot x[k]) = 2$$

6. **Step**  $n = n + 1 = 2$

7. **Calculate**

$$y[2] = \sum (h[1 - k] \cdot x[k]) = 3$$

8. **Step**  $n = n + 1 = 3$

9. **Calculate**

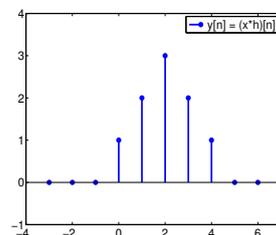
$$y[3] = \sum (h[1 - k] \cdot x[k]) = 2$$

10. **Step**  $n = n + 1 = 4$

11. **Calculate**

$$y[4] = \sum (h[1 - k] \cdot x[k]) = 1$$

12. **Stop**, because overlap ends.



**Example:**

1. **Flip**

$$h[-k]$$

2. **Shift**  $h[n - k]$  (by changing  $n$ ) to where overlap begins

$$h[-3 - k]$$

(i.e., overlap begins at  $n = -3$ ).

$$n = -3$$

3. **Calculate**

$$y[-3] = \sum (h[-3 - k] \cdot x[k]) = -1$$

4. **Step**  $n = n + 1 = -2$

5. **Calculate**

$$y[-2] = \sum (h[-2 - k] \cdot x[k]) = -1$$

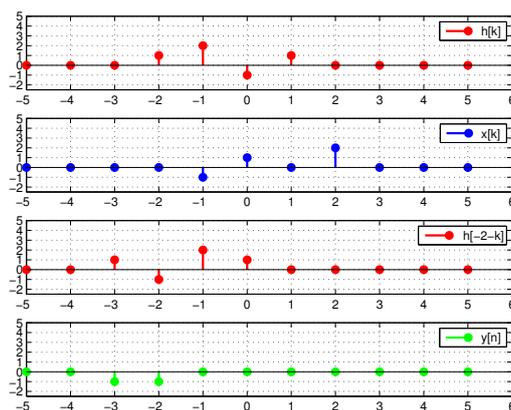
6. **Step**  $n = n + 1 = -1$

7. **Calculate**

$$y[0] = \sum (h[0 - k] \cdot x[k]) = 3$$

8. **Step**  $n = n + 1 = 0$

and so on . . . , until overlap ends.



Notice that the result  $y[n]$  begins (i.e., is magnitude  $y[n] \neq 0$  for the first time) at  $n = -3$ . This represents the starting point  $n$  where  $x[k]$  and the shifted  $h[n - k]$  first intersect (specifically, at  $n = -3$ ).

**(CT) Convolution steps**

To convolute  $x(t) * h(t)$ ; first, write both as functions of  $\tau$ ,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau$$

then

1. **Flip**  $h(\tau)$  (to  $h(-\tau)$ )  $t$  is where  $\tau = 0$  label the piecewise parts of the function (e.g.,  $t - 1$  at  $\tau = -1$ , etc.)
2. **Shift**  $h(t - \tau)$  from where  $t = -\infty$  until the  $t$  where the functional form changes (e.g., where overlap between  $h(t - \tau)$  and  $x(\tau)$  begins/changes)
3. While  $t < \infty$ :
  - **Shift**  $h(t - \tau)$  to the right (i.e., from  $t = -\infty, \dots$ ) until the functional form changes. Record  $t$  at this point.
4. Note the  $t$ -intervals discovered, and, for each of these intervals:
  - (a) **Pick** and label a random point in the interval (e.g., in the middle)
  - (b) **Draw**  $h(t - \tau)$ , with the piecewise labels from above
  - (c) **Shade** the overlapping area between  $x(\tau)$  and  $h(t - \tau)$ . Determine the upper and lower bounds of the shaded area (these are the bounds for the integral)
5. **Solve the integrals**

We can imagine CT convolution as:

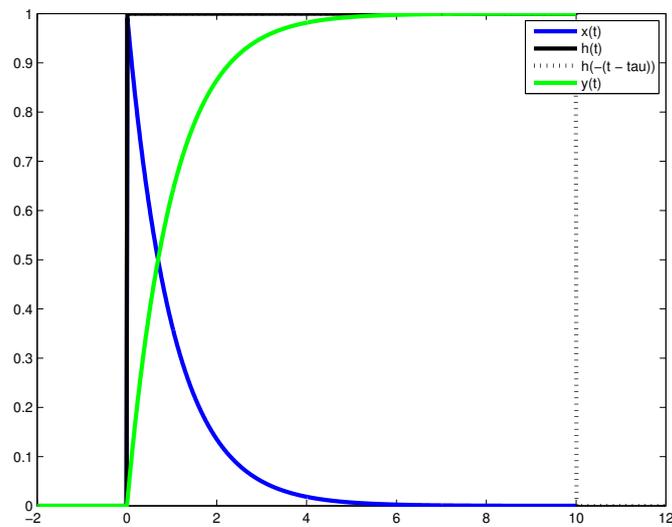
$$\begin{aligned} y(t) &= \lim_{\Delta \rightarrow 0} \sum_{k=-\infty}^{\infty} x(k\Delta)p(t - k\Delta)\Delta \\ &= \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau \end{aligned}$$

where  $p(t)$  is the **pulse** (Eq. 4) of width  $\Delta$ . As  $\Delta \rightarrow 0$ ,  $k\Delta \rightarrow \tau$ ,  $\Delta \rightarrow d\tau$ ,  $p(t) \rightarrow \delta(t)$ . and thus Eq. 9.

Example: Compute  $y(t) = h(t) * u(t)$  for the system:

$$u(t) \rightarrow \boxed{h(t) = e^{-at}u(t)} \rightarrow y(t) = h(t) * u(t)$$

$$\begin{aligned} y(t) &= h(t) * u(t) \\ &= \int_{-\infty}^{+\infty} h(\tau)u(t - \tau)d\tau \\ &= \int_0^t e^{-a\tau}d\tau \\ &= \frac{1}{-a} [e^{-a\tau}]_0^t \\ &= \frac{1}{-a} (e^{-at} - e^{-a0}) \\ &= \frac{-1}{a} (e^{-at} - 1) \\ &= \frac{1}{a} [1 - e^{-at}] \\ &= \begin{cases} \frac{1}{a} [1 - e^{-at}] & t > 0 \\ 0 & t < 0 \end{cases} \end{aligned}$$



Note that the amplitude of  $y(t)$  approaches  $\frac{1}{a}$  (where  $a = 1$  in this plot).

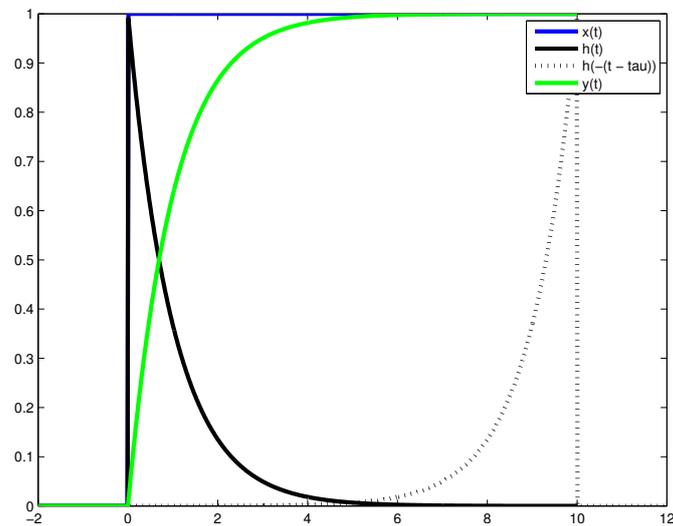
☞ **Example:** Compute  $y(t) = u(t) * h(t)$  for the system

$$u(t) \rightarrow \boxed{h(t) = e^{-at}u(t)} \rightarrow y(t) = h(t) * u(t)$$

(identical to the previous example, but we will convolute  $u(t)$  with  $h(t)$  instead of vice versa).

$$\begin{aligned} y(t) &= u(t) * h(t) \\ &= \int_{-\infty}^{+\infty} u(\tau)h(t - \tau)d\tau \\ &= \int_0^t 1 \cdot u(\tau)e^{-a(t-\tau)}d\tau \\ &= \int_0^t 1 \cdot e^{-at}e^{a\tau}d\tau \\ &= \frac{1}{a}e^{-at} \left[ e^{a\tau} \right]_0^t \\ &= \frac{1}{a}e^{-at}(e^{at} - 1) \\ &= \frac{1}{a}(e^{-at-at} - e^{-at}) \\ &= \frac{1}{a}(1 - e^{-at}) \\ &= \begin{cases} \frac{1}{a}[1 - e^{-at}] & t > 0 \\ 0 & t < 0 \end{cases} \end{aligned}$$

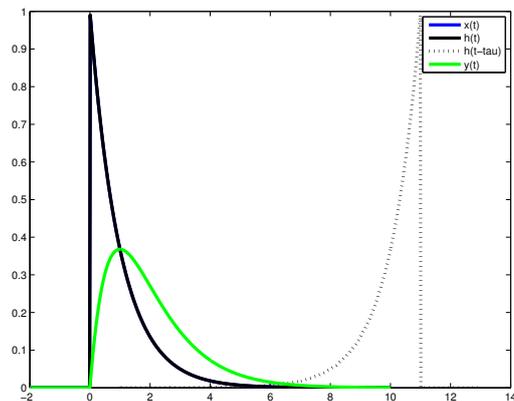
which is the same!



Example: Compute the convolution  $y(t) = u(t) * h(t)$  where

$$y(t) = x(t) * h(t) = (e^{-t}u(t)) * (e^{-t}u(t))$$

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{+\infty} (e^{-\tau}u(\tau)) * (e^{-(t-\tau)}u(t-\tau))d\tau \\ &= \int_0^t e^{-\tau}e^{-(t-\tau)}d\tau \\ &= \int_0^t e^{-\tau}e^{-t+\tau}d\tau \\ &= \int_0^t e^{-\tau}e^{-t}e^{\tau}d\tau \\ &= \int_0^t e^0e^{-t}d\tau \\ &= e^{-t} \int_0^t 1d\tau \\ &= e^{-t} \cdot t \\ &= te^{-t} \\ &= \begin{cases} te^{-t} & t > 0 \\ 0 & t < 0 \end{cases} \end{aligned}$$



**Example:** We have two boxes of the same size and position: both from  $-0.5$  to  $+0.5$  (width 1) and with height 1. To convolute these two  $x(t) * h(t)$ ;

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau$$

First, write and graph both as functions of  $\tau$ :  $x(\tau)$  and  $h(\tau)$ . Then

1. **Flip**  $h(\tau)$  (to  $h(-\tau)$ )
2. **Draw**  $h(t - \tau)$  some distance to the left of  $x(\tau)$ , i.e., where  $t$  is small. Label the left side  $\tau = -1+t$ , and right side at  $\tau = 0.5+t$ .
3. While  $t < \infty$ :
  - **Shift**  $h(t - \tau)$  to the right (i.e., from  $t = -\infty, \dots$ ) until the functional form changes. Record  $t$  at this point.
4. Set (using the  $t$  intervals we discovered):

$$y(t) = \begin{cases} 0 & t \leq -1 \\ ? & -1 \leq t \leq 0 \\ ? & 0 \leq t \leq 1 \\ 0 & t > 1 \end{cases}$$

We may now draw the zero parts (where there is no overlap). For the remaining intervals:

- (a) **Pick** and label a random point in the interval (e.g., in the middle)
  - (b) **Draw**  $h(t - \tau)$  and label the side which falls in the interval
  - (c) **Shade** the overlapping area between  $x(\tau)$  and  $h(t - \tau)$ . Determine the upper and lower bounds of the shaded area (these are the bounds for the integral)
5. **Set**

$$y(t) = \begin{cases} 0 & t \leq -1 \\ \int_{-0.5}^{t+0.5} 1d\tau = t + 1 & -1 \leq t \leq 0 \\ \int_{t-0.5}^{0.5} 1d\tau = -t + 1 & 0 \leq t \leq 1 \\ 0 & t > 1 \end{cases}$$

**Convolution as a filter** We can imagine convolution not only as a black box, but as a **filter**: We input a stream of data ( $x[n]$ ) into the filter and obtain an output stream  $y[n]$ .

Convolution is

- an important computational tool, and
- an important conceptual tool.

### 3.6 Interconnection of LTI Systems

**Convolution properties**

- **Commutative**  $x(t) * h(t) = h(t) * x(t)$
- **Associative**  $[x(t) * g(t)] * h(t) = x(t) * [g(t) * h(t)]$
- **Distributive**  $x(t) * [g(t) * h(t)] = x(t) * g(t) + x(t) * h(t)$

The **commutative** property means that

$$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$$

is equivalent to

$$h[n] \rightarrow \boxed{x[n]} \rightarrow y[n]$$

The **associative property** means that

$$x[n] \rightarrow \boxed{h_1[n]} \rightarrow \boxed{h_2[n]} \rightarrow y[n]$$

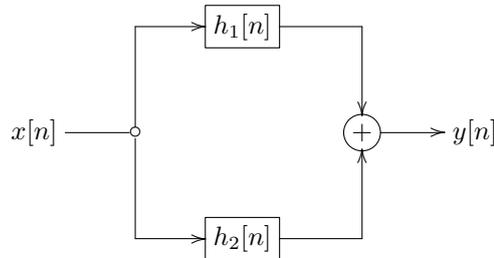
is equivalent to

$$x[n] \rightarrow \boxed{h_1[n] * h_2[n]} \rightarrow y[n]$$

and

$$x[n] \rightarrow \boxed{h_2[n]} \rightarrow \boxed{h_1[n]} \rightarrow y[n]$$

The **distribute property** means that



is equivalent to

$$x(t) \rightarrow \boxed{h_1(t) + h_2(t)} \rightarrow y(t)$$

as we can also see by:

$$\begin{aligned} &= x(t) * (h_1(t) + h_2(t)) \\ &= \int x(\tau)(h_1(t - \tau) + h_2(t - \tau)d\tau) \\ &= \int x(\tau)h_1(t - \tau) + x(\tau)h_2(t - \tau)d\tau \\ &= \int x(\tau)h_1(t - \tau) + \int x(\tau)h_2(t - \tau)d\tau \end{aligned}$$

**Invertible Systems and Deconvolution** If we invert an  $h(t)$ , and form an identity system:

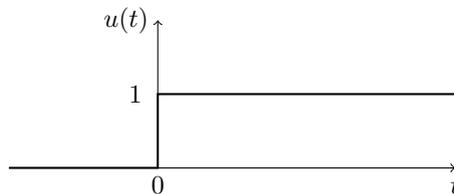
$$x(t) \rightarrow \boxed{h(t)} \rightarrow \boxed{h_i(t)} \rightarrow x(t)$$

(where  $h_i(t)$  is the *inverse* of  $h(t)$ ) we note that:

$$h(t) * h_i(t) = \delta(t)$$

### 3.7 (Unit) step response to an LTIS

In practice, impulses are difficult to generate and work with, so it can be difficult to study an LTIS with impulse responses. We're often interested in the response of an LTIS to the **unit step** (refer to Section 2.7.1) or any step. Recall:



Like if we *step up* the temperature in an air-conditioning system, the voltage in an electrical circuit, the dosage of a drug to a patient, etc.

$$\underbrace{u(t)}_{\text{unit step}} \rightarrow \boxed{\text{LTIS}} \rightarrow \underbrace{y(t)}_{\text{response}}$$

The reaction to the unit step is important. The output signal will change. One of three things is likely to happen:

1. The system goes out of control (i.e., in an **unstable** system)
2. The output signal converges on some new value
3. The output signal oscillates around some new value

We are interested in

- **steady state error (SSE)** the error between the target step up, and the actual signal
- **rise time** how fast to get from 10 percent of target value to 90 percent
- **overshoot** (e.g., in percent): how far does the signal initially overshoot
- **settling time** the time until the error settles below some specified error threshold (e.g., in percent)

We often want to improve one of these, and usually it comes as a trade off with the others. For example, to improve the rise time involved with the effect of a certain drug on a patient, we risk increasing increasing overshoot.

From the convolution integral:

$$\begin{aligned} y(t) &= \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau \\ &= \int_{-\infty}^{+\infty} u(\tau)h(t-\tau)d\tau \\ &= \int_0^t 1 \cdot h(t-\tau)d\tau \end{aligned}$$

If the system is **causal**, then  $h(t-\tau)$  is zero for  $(t-\tau) < 0$  and for  $\tau > t$ , and

$$y(t) = \int_0^t h(\tau)d\tau \quad (11)$$

where  $y(t)$  is the **unit step response**. It can be calculated directly from the impulse response!

Differentiating,

$$h(t) = \frac{dy(t)}{dt} \quad (12)$$

we see that the unit impulse can be calculated directly from the unit step response (and thus completely describes the input-output characteristic of an LTIS).

**Example:** Getting a step response from the impulse response.

Consider

$$h(t) = e^{-3t}u(t)$$

This system is causal. Thus, from Eq. 11,

$$\begin{aligned} y(t) &= \int_0^t h(\tau)d\tau \\ &= \int_0^t e^{-3\tau}d\tau \\ &= \left[ \frac{e^{-3\tau}}{-3} \right]_0^t \\ &= \left( \frac{-e^{-3t}}{-3} - \frac{1}{-3} \right) u(t) \\ &= \frac{1}{3}(1 - e^{-3t})u(t) \end{aligned}$$

We can verify this, with Eq. 12:

$$\begin{aligned} h(t) &= \frac{d}{dt}y(t) \\ &= \frac{d}{dt} \frac{1}{3}(1 - e^{-3t})u(t) \\ &= \frac{1}{3}(1 - e^{-3t})\delta(t) + \frac{1}{3}(-e^{-3t})(-3)u(t) \\ &= -(-e^{-3t})u(t) \\ &= e^{-3t}u(t) \end{aligned}$$

## 4 Fourier Series

## 4.1 Response of LTI Systems to Complex Exponentials

So far, we studied LTI systems because

- they can be used to construct a broad and useful class of signals, and
- the response to an LTIS provides us with a convenient representation.

Previously, we used shifted unit impulses as basic signals. Now, we will use complex exponentials. Both advantages (above) are provided by the set of **complex exponentials** (recall, Eq. 5 in Section 2.7.2):

$$x(t) = e^{at}$$

with complex number  $a$  (or  $Ce^{at}$  with complex numbers  $C$  and  $a$ ). Note that the LTIS changes the amplitude of this signal only! Thus,

$$x(t) \rightarrow \boxed{\text{LTIS}} \rightarrow \lambda x(t)$$

where  $x(t)$  is a complex exponential. In this case, it is a **complex amplitude**, a function of  $a$ :

$$e^{at} \rightarrow \boxed{\text{LTIS}} \rightarrow H(a)e^{at} \quad (13)$$

i.e.,  $H(a)$ . This system is composed as follows:

$$\underbrace{x(t) \rightarrow \boxed{\text{LTIS}} \rightarrow \underbrace{\lambda}_{\text{eigen value}} x(t)}_{\text{eigen function}}$$

where

- an **eigen function** is a signal for which the system output is just a constant times the input (a scalar multitude)  $x(t)$ ;
- the **eigen value** is the amplitude factor (the multiplier)  $\lambda$ .

We can derive this relation with convolution:

$$\begin{aligned} y(t) &= h(t) * x(t) \\ &= \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau \\ &= \int_{-\infty}^{+\infty} h(\tau)e^{a(t-\tau)}d\tau \\ &= \int_{-\infty}^{+\infty} h(\tau)e^{at-a\tau}d\tau \\ &= \int_{-\infty}^{+\infty} h(\tau)e^{at}e^{-a\tau}d\tau \\ &= e^{at} \int_{-\infty}^{+\infty} h(\tau)e^{-a\tau}d\tau \\ &= \underbrace{H(a)}_{\lambda} \underbrace{e^{at}}_{\text{signal}} \end{aligned}$$

and thus Eq. 13 with complex amplitude. In other words the response of an LTIS to a **complex exponential** is the same complex exponential, but with a change in amplitude; i.e., *all* complex exponentials are **eigen functions** of LTI systems:

We know that we can construct complicated signals by summing unit sample signals (see, for example, Fig. 4). We could similarly start with a family of **exponentials** or **sinusoidal primitives** (i.e., simple sine and cosine – either continuous or discretely sampled) and construct more complicated signals.

**Important property of sinusoids in an LTIS**  
 If the input is an eternal sinusoid, e.g.,

$$x(t) = \cos(\omega t)$$

the output will be an eternal sinusoid, e.g.,

$$y(t) = A \cos(\omega t + \phi)$$

with

- *same* **frequency**  $\omega$
- different **amplitude**  $A$  (may be bigger or smaller)
- different **phase angle**  $\phi$  (may be slightly delayed)

Whereas in time analysis we broke a signal into a sum of weighted, shifted, impulse responses, in **frequency analysis** (hence the use of  $\omega$ ) we break a signal into a sum of sinusoids of different frequencies. We can do this on *any* signal that has finite energy.

Let  $x(t)$  correspond to a linear combination of three complex exponentials (where, e.g.,  $a = \omega j$ ):

$$x(t) = C_1 e^{a_1 t} + C_2 e^{a_2 t} + C_3 e^{a_3 t}$$

The **response** to each separately is (for  $k = 1, 2, 3$ ):

$$C_k e^{a_k t} \rightarrow \boxed{\text{LTIS}} \rightarrow C_k H(a_k) e^{a_k t}$$

i.e., the response is the sum of responses:

$$y(t) = C_1 H(a_1) e^{a_1 t} + C_2 H(a_2) e^{a_2 t} + C_3 H(a_3) e^{a_3 t}$$

More generally:

$$\sum_k C_k e^{a_k t} \rightarrow \boxed{\text{LTIS}} \rightarrow \sum_k C_k H(a_k) e^{a_k t}$$

Thus, if we know the eigen values  $H(a_k)$  then the response to a linear combination of complex exponentials can be constructed in a straightforward manner; An LTIS can be characterized entirely by how it treats sinusoids of different frequencies  $\omega$ .

**The transfer function**  $H(\omega)$

Defines how inputs *transfer* to outputs:

$$Y(\omega) = H(\omega)X(\omega)$$

so,

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

Expressed in polar (magnitude-phase) form,

$$H(\omega) = |H(\omega)|e^{j\phi(\omega)}$$

and thus

$$\begin{aligned} H(\omega)e^{j2\pi\omega t} &= |H(\omega)|e^{j\phi(\omega)}e^{j2\pi\omega t} \\ &= |H(\omega)|e^{j(2\pi\omega t + \phi(\omega))} \end{aligned}$$

1.  $|H(\omega)|$  is the **modulation transfer function** (MTF)
2.  $\phi(\omega)$  is the **phase transfer function** (PTF)

$H(\omega)$  will be important when looking at the **Laplace transform** and **Z transform**. It can be derived from differential equations.

## 4.2 Introduction to Fourier Series and Frequency Analysis

In **frequency analysis**, we decompose signals into sinusoids of different **frequencies**. The famous Fourier put forth the idea that *any periodic function can be represented by a series of sines and cosines* which are harmonically related. This has many applications, including:

- Musical analysis
- Vibrating strings
- Orbits of planets
- Radiation
- Cell phones must use different frequencies
- Speech recognition / compression
- Filtering: boosting specific frequencies
- Analysis of linear systems
- Radio: Each station has it's own frequency

Now we will look at **Fourier analysis**, using the concepts we have been working with so far (see Section A):

- Transforms

- Taylor series
- Sinusoids
- Exponentials and complex exponentials
- Euler's relations

Fourier analysis is a type of **frequency domain analysis**, as opposed to the time domain which we have been working with previously;

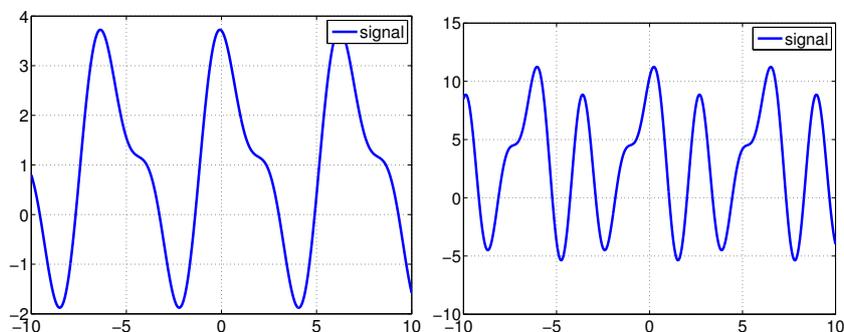
- time domain  $x(t)$
- frequency domain  $|x(\omega)|$

In a LTIS, we might want to change the **frequency response**. For example, by manipulating controls in an audio system for more **bass** (low frequency) or **treble** (high frequency). Or, in analysis of the stock market, we may want to look at only weekly movements (high frequency), and ignore the boom/bust cycles (low frequency), or vice versa.

In frequency analysis we deal with **harmonics**, analogous to in music. Similar to what we saw above with sinusoids, note that combining the  $k$ -th and  $l$ -th **harmonic**,

$$e^{jk\omega_0 t} \cdot e^{jl\omega_0 t} = e^{j(k+l)\omega_0 t}$$

produces another harmonic ( $k+l$ ) with the *same* fundamental **frequency**! With music instruments, the same note produces the same **fundamental frequency**, but sounds different due to the other harmonics. For example, these



could come from two different instruments playing the same note (fundamental frequency  $\omega_0 = 1$ ;  $T_0 = 2\pi$ ).

### 4.3 Fourier Series Representation of CT Periodic Signals: Analysis and Synthesis Equations

Almost any **periodic** signal can be represented by a Fourier series.

Two instruments playing the same note (i.e., at the same fundamental frequency), sound different because of the different **harmonics**. The **Fourier series** will break the sound into its frequency and harmonic (corresponding to phase and magnitude).

### How to determine a Fourier Series Representation

1. (Optional) Confirm that it is **periodic** (otherwise, use a FT!)
2. Find the **fundamental period**  $T_0$
3. Rewrite as exponentials using **Euler's formula**
4. Reduce and collect terms

We represent signal  $x(t)$  as

### Synthesis Equation

$$x(t) = \sum_{k=-\infty}^{k=+\infty} C_k e^{jk\omega_0 t} \quad (14)$$

$$C_k = C_{-k}^*$$

The frequency  $\omega_0$  is the **fundamental frequency** aka **first harmonic**. The  $k\omega_0$  is called the  **$k$ -th harmonic**. The  $\{C_k\}$ s are the **Fourier series coefficients**; where

### Analysis Equation

$$C_k = \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt \quad (15)$$

and from here (Eq. 15), we get for  $k = 0$ ,

$$C_0 = \frac{1}{T_0} \int_{T_0} x(t) dt$$

(the average value of  $x(t)$  over *one period*  $T_0$ ). The analysis equation shows: how much of the  $k$ -th component there is. Note how  $x(t)$  from the synthesis equation plugs into the analysis equation; and vice versa with respect to  $C_k$ . Each  $C_k$  is a **complex number**, representable in either Cartesian form,

$$C_k = a + jb$$

or polar / Euler form

$$C_k = |C_k| e^{j\theta_k}$$

(with magnitude  $|C_k|$  and phase  $\theta_k = \angle C_k$ ). Since  $C_{-k} = C_k^*$  it follows that  $\theta_{-k} = -\theta_k$ . For simple cases (e.g., square wave, triangle wave) there are analytical solutions.

The sum of two terms of the same frequency  $k\omega_0$  gives

$$\begin{aligned} C_{-k} e^{-jk\omega_0 t} + C_k e^{jk\omega_0 t} &= |C_k| e^{-j\theta_k} e^{-jk\omega_0 t} + |C_k| e^{j\theta_k} e^{jk\omega_0 t} \\ &= |C_k| \left[ e^{-j(k\omega_0 t + \theta_k)} + e^{j(k\omega_0 t + \theta_k)} \right] \\ &= 2|C_k| \cos(k\omega_0 t + \theta_k) \end{aligned}$$

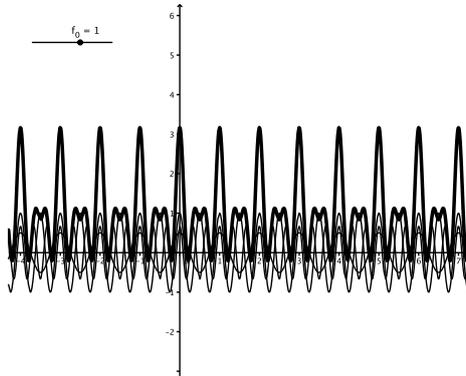


Figure 5: Construction of a signal from a linear combination of periodic sinusoids  $x(t) = x_1(t) + x_2(t) + \cdots + x_5(t)$ .

and thus, given the Fourier coefficients  $C_k$ , we can easily get the **Fourier series trigonometric form**:

$$\begin{aligned}
 x(t) &= C_0 + \sum_{k=1}^{\infty} 2|C_k| \cos(k\omega_0 t + \theta_k) \\
 &= C_0 + \sum_{k=1}^{\infty} [A_k \cos k\omega_0 t + B_k \sin k\omega_0 t]
 \end{aligned}$$

Example: Find the Fourier Series representation for the signal:

$$x(t) = \sin \omega_0 t$$

Periodic with fundamental period  $T_0 = \frac{2\pi}{\omega_0}$ . We can rewrite as exponentials using **Euler's relations**; reduce and collect terms:

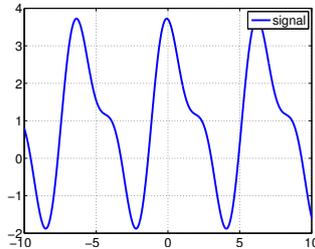
$$\begin{aligned} x(t) &= \sin \omega_0 t \\ &= \frac{1}{2j} [e^{j\omega_0 t} - e^{-j\omega_0 t}] \\ &= \underbrace{\frac{1}{2j}}_{C_1} e^{j\omega_0 t} - \underbrace{\frac{1}{2j}}_{C_{-1}} e^{-j\omega_0 t} \\ &= \sum_{k=-1}^{k=+1} C_k e^{jk\omega_0 t} \quad (\text{The synthesis equation, Eq. 14}) \end{aligned}$$

where:

- $C_0 = 0$
- $C_1 = \frac{1}{2j}$  ;
- $C_{-1} = -\frac{1}{2j}$ ; and
- $C_{k \neq 1, -1} = 0$  (all other coefficients).

**Example:** Consider the signal

$$x(t) = 1 + \sin \omega_0 t + 2 \cos \omega_0 t + \cos(2\omega_0 t + \frac{\pi}{4})$$



which is periodic with period  $T_0 = \frac{2\pi}{\omega_0}$ . Using **Euler's relation** (see Section A.2), we can manipulate the signal:

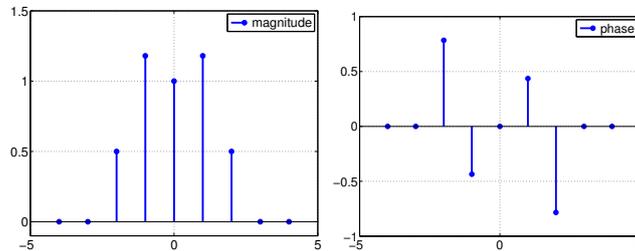
$$\begin{aligned} x(t) &= 1 + \sin \omega_0 t + 2 \cos \omega_0 t + \cos(2\omega_0 t + \frac{\pi}{4}) \\ &= 1 + \frac{1}{2j}[e^{j\omega_0 t} - e^{-j\omega_0 t}] + [e^{j\omega_0 t} - e^{-j\omega_0 t}] + \frac{1}{2}[e^{j(2\omega_0 t + \frac{\pi}{4})} + e^{-j(2\omega_0 t + \frac{\pi}{4})}] \\ &= 1 + \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t} + e^{j\omega_0 t} - e^{-j\omega_0 t} + \frac{1}{2}e^{j(2\omega_0 t + \frac{\pi}{4})} + \frac{1}{2}e^{-j(2\omega_0 t + \frac{\pi}{4})} \\ &= 1 + (1 + \frac{1}{2j})e^{j\omega_0 t} - (1 + \frac{1}{2j})e^{-j\omega_0 t} + (\frac{1}{2}e^{j\frac{\pi}{4}})e^{j2\omega_0 t} + (\frac{1}{2}e^{-j\frac{\pi}{4}})e^{-j2\omega_0 t} \end{aligned}$$

where, in **Cartesian form** ( $a + bj$ ) or **Euler / polar form** ( $|C|e^{j\theta}$ ),

	$C_k$	$C_{-k}$
$k = 0$	1	N/A
$k = 1$	$1 + \frac{1}{2j}$	$1 - \frac{1}{2j}$
$k = 2$	$\frac{1}{2}e^{j(\frac{\pi}{4})} = \frac{\sqrt{2}}{4}(1 + j)$	$\frac{1}{2}e^{-j(\frac{\pi}{4})} = \frac{\sqrt{2}}{4}(1 - j)$
$k \geq 3$	0	0

which we can plot as:

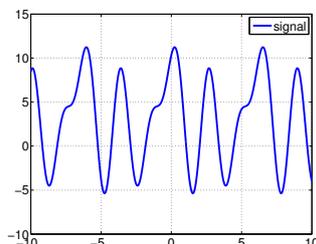
- **magnitude**  $|C_k|$ , e.g.,  $|C_1| = \sqrt{1^2 + 0.5^2} \approx 1.1180$ ; and
- **phase**  $\angle C_k$ , e.g.,  $\angle C_1 = \phi(1 + 0.5j) = 0.4360$



Note that these are even and odd functions. This is all the information we need to reconstruct the signal.

**Example:** Consider the signal

$$x(t) = 3 + 3 \cos \omega_0 t + 5 \cos(2\omega_0 t + \frac{\pi}{6}) + 4 \sin 3\omega_0 t$$



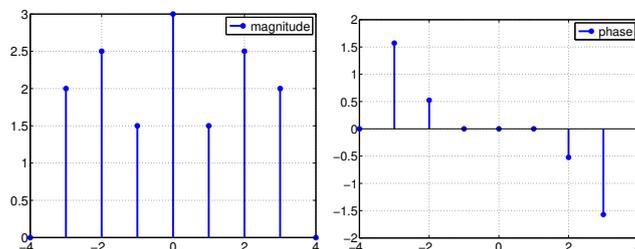
which is periodic with period  $T_0 = \frac{2\pi}{\omega_0}$ . Using **Euler's relation** (see Section A.2), we can manipulate the signal:

$$\begin{aligned} x(t) &= 3 + 3 \cos \omega_0 t + 5 \cos(2\omega_0 t + \frac{\pi}{6}) + 4 \sin 3\omega_0 t \\ &= 3 + 3 \frac{1}{2} [e^{j\omega_0 t} + e^{-j\omega_0 t}] + 5 \frac{1}{2} [e^{2\omega_0 t + \frac{\pi}{6}} - e^{-j(2\omega_0 t + \frac{\pi}{6})}] + 4 \frac{1}{2j} [e^{j3\omega_0 t} - e^{-j3\omega_0 t}] \\ &= 3 + \frac{3}{2} [e^{j\omega_0 t} + e^{-j\omega_0 t}] + \frac{5}{2} [e^{2\omega_0 t + \frac{\pi}{6}} - e^{-j(2\omega_0 t + \frac{\pi}{6})}] + \frac{4}{2j} [e^{j3\omega_0 t} - e^{-j3\omega_0 t}] \\ &= 3 + \frac{3}{2} e^{j\omega_0 t} - \frac{3}{2} e^{-j\omega_0 t} + \frac{5}{2} e^{2\omega_0 t + \frac{\pi}{6}} - \frac{5}{2} e^{-j(2\omega_0 t + \frac{\pi}{6})} + \frac{4}{2j} e^{j3\omega_0 t} - \frac{4}{2j} e^{-j3\omega_0 t} \\ &= C_0 + C_1 e^{j1\omega_0 t} - C_1 e^{-j1\omega_0 t} + C_2 e^{2\omega_0 t} - C_2 e^{-j2\omega_0 t} + C_3 e^{j3\omega_0 t} - C_3 e^{-j3\omega_0 t} \\ &= \sum_{k=-3}^{k=+3} C_k e^{jk\omega_0 t} \end{aligned}$$

where

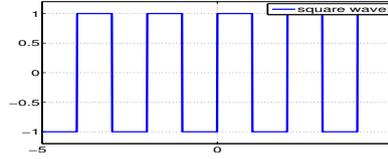
	$C_k$	$C_{-k}$
$k = 0$	3	—
$k = 1$	1.5	1.5
$k = 2$	$2.5 \angle \frac{\pi}{6}$	$2.5 \angle -\frac{\pi}{6}$
$k = 3$	$2 \angle \frac{\pi}{2}$	$2 \angle -\frac{\pi}{2}$
$k \geq 4$	0	0

in **polar form** (**magnitude**  $\angle$  **phase**) form, where, note, the  $k = 3$ -rd coefficient lies on purely on the imaginary axis  $\frac{\pi}{2}$ .



Note how the  $C_k$  contain the phase angle (e.g.,  $+\frac{\pi}{6}$ ).

 **Example:** The periodic square wave:



Note that it is periodic:  $T_0 = 2$ ;  $\omega_0 = \frac{2\pi}{T_0} = \pi$ . We use the **analysis equation** Eq. 15 to find:

$$\begin{aligned}
 C_0 &= \frac{1}{T_0} \int_{T_0} x(t) dt \\
 &= \frac{1}{2} \int_0^2 x(t) dt \\
 &= \frac{1}{2} \left[ \int_0^1 1 dt + \frac{1}{2} + \int_1^2 -1 dt \right] \\
 &= \frac{1}{2} [1 - 1] \\
 &= 0
 \end{aligned}$$

and

$$\begin{aligned}
 C_k &= \frac{1}{T_0} \int_{T_0} x(t) e^{-jk\omega_0 t} dt \\
 &= \frac{1}{2} \int_0^2 x(t) e^{-jk\pi t} dt \\
 &= \frac{1}{2} \int_0^1 e^{-jk\pi t} dt - \frac{1}{2} \int_1^2 e^{-jk\pi t} dt \\
 &= \frac{1}{2} \left[ \frac{1}{-jk\pi} e^{-jk\pi t} \right]_0^1 - \frac{1}{2} \left[ \frac{1}{-jk\pi} e^{-jk\pi t} \right]_1^2 \\
 &= \frac{1}{2} \frac{1}{-jk\pi} \left[ e^{-jk\pi} - 1 \right] - \frac{1}{2} \frac{1}{-jk\pi} \left[ e^{-2jk\pi} - e^{-jk\pi} \right] \\
 &= \frac{1}{-j2k\pi} \left[ e^{-jk\pi} - 1 \right] + \frac{1}{j2k\pi} \left[ e^{-2jk\pi} - e^{-jk\pi} \right] \\
 &= \frac{1}{-j2k\pi} \left[ e^{-jk\pi} - 1 \right] + \frac{1}{j2k\pi} \left[ 1 - e^{-jk\pi} \right] \\
 &= \begin{cases} \frac{-1}{-j2k\pi} \left[ -1 - 1 \right] + \frac{1}{j2k\pi} \left[ 1 - -1 \right] & k \text{ is odd} \\ \frac{-1}{-j2k\pi} \left[ 1 - 1 \right] + \frac{1}{j2k\pi} \left[ 1 - 1 \right] & k \text{ is even} \end{cases} \\
 &= \begin{cases} \frac{2}{jk\pi} & k \text{ is odd} \\ 0 & k \text{ is even} \end{cases}
 \end{aligned}$$

Fig. 6 is drawn for  $k = 0, \pm 1, \dots, \pm 7$ .

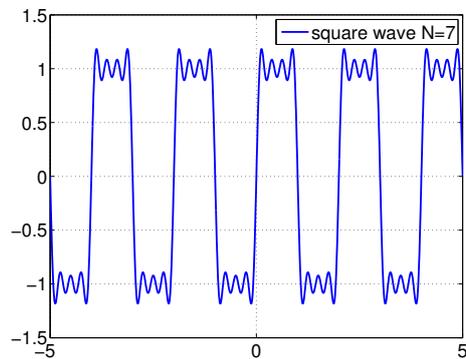


Figure 6: Approximation of the square wave, using  $N = 7$ . As we increase  $N$  for  $k \pm N$ , it will converge. Already we see some convergence. This is an illustration of the **Gibbs phenomenon**.

☞ **Note:** Recall that (via **Euler**):

$$\begin{aligned} e^{-j2k\pi} &= \cos k2\pi - j \sin 2k\pi \\ &= 1 - j0 \\ &= 1 \end{aligned}$$

for any  $k = \dots, -4, -2, 0, 2, 4, \dots$

☞ **Note:** Recall that (via **Euler**):

$$e^{-jk\pi} = \cos k\pi - j \sin k\pi = \begin{cases} 1 - 0 & k \text{ is even} \\ 0 - 1 & k \text{ is odd} \end{cases} = \begin{cases} 1 & k \text{ is even} \\ -1 & k \text{ is odd} \end{cases}$$

since  $\cos(k\pi) = 1$  for any *even*  $k$ , and  $= -1$  for any *odd*  $k$ ; And  $\sin k\pi = 0$  for any integer  $k$ ; and  $= 1$  otherwise.

#### 4.4 Convergence

We can approximate Eq. 14 to some  $N$ :

$$x_N(t) = \sum_{k=-N}^N C_k e^{jk\omega_0 t}$$

i.e., approximating  $x(t)$  with a *finite* linear combination of harmonically-related complex exponentials.

Fig. 6 plots  $x_N$  for  $N = 7$ . Using more coefficients (higher  $N$ ) leads to a better approximation. Fig. 7 plots the magnitude  $|C_k|$  and angle  $\angle C_k$ .

Let  $e_N$  be the approximation error:

$$e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^N C_k e^{jk\omega_0 t}$$

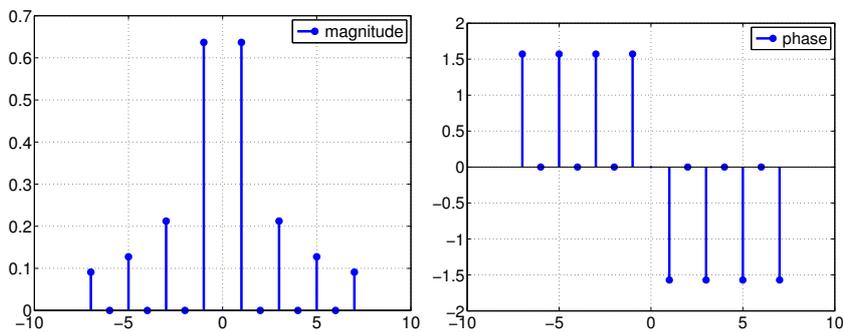


Figure 7: The magnitude  $|C_k|$  and phase/frequency  $\angle C_k$  plots of the square wave example with  $k = -7, -6, \dots, 0, \dots, 6, 7$ .

We can determine error as the measure the energy of  $e_N(t)$  over one period:

$$E_N = \int_{T_0} |e_N(t)|^2 dt$$

The particular choice of coefficients to minimize the energy of the error is given by Eq. 15 (the exact expression used to determine the Fourier series coefficients).

## 4.5 Properties of the CT Fourier Series

1. Linearity
2. Time invariance
  - (a) time shifting
  - (b) time reversal
  - (c) time scaling

Lets say that the notation  $x(t) \rightarrow_{FS} C_k$  expresses the relationship between Eq. 15 and Eq. 14 in a Fourier series of periodic signal  $x(t)$  with period  $T$  and fundamental frequency  $\omega_0$ .

**Linearity** If  $x(t) \leftrightarrow_{FS} C_k$  and  $y(t) \leftrightarrow_{FS} D_k$  then we have

$$z(t) = \alpha x(t) + \beta(y(t) \leftrightarrow_{FS} E_k = \alpha C_k + \beta D_k$$

**Time Shifting** If  $x(t) \leftrightarrow_{FS} C_k$  then

$$x(t - t_d) \leftrightarrow_{FS} e^{-jk\omega_0 t} C_k$$

and the magnitudes of its Fourier series coefficients remain unchanged.

**Time Reversal** If  $x(t) \leftrightarrow_{FS} C_k$  then

$$x(-t) \leftrightarrow_{FS} C_{-k}$$

**Time Scaling** If  $x(t) = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 t}$ , then

$$x(\alpha t) = \sum_{k=-\infty}^{\infty} C_k e^{jk(\alpha\omega_0)t}$$

The FS coefficients have not changed, only the FS representation has changed because of the change in fundamental frequency.

**Multiplication** For two periodic signals  $x(t)$  (with FS coefficients  $C_k$ ) and  $w(t)$  (with FS coefficients  $D_k$ ) with period  $T$ , since the product  $x(t)y(t)$  is also periodic with the same period  $T$ ,

$$x(t)y(t) \leftrightarrow_{FS} E_k = \sum_{l=-\infty}^{\infty} C_l D_{k-l}$$

where the right-hand side can be seen as the discrete-time convolution of the sequence representing the Fourier coefficients of  $x(t)$  and the sequence representing the Fourier coefficients of  $y(t)$ .

**Conjugate and Conjugate Symmetry** If  $x(t) \leftrightarrow_{FS} C_k$ ; then

$$x^*(t) \leftrightarrow_{FS} C_k^*$$

and if  $x(t)$  is real, i.e.,  $x(t) = x^*(t)$ , the FS coefficients will be **conjugate symmetric**, i.e.,

$$C_{-k} = C_k^*$$

## 4.6 FS Representations of DT Periodic Signals

As in CT systems,  $x[n] = e^{j\omega n}$  is an eigenfunction of the system, with output

$$y[n] = H(e^{j\omega})e^{j\omega n} \quad (16)$$

with eigenfunction  $e^{j\omega n}$  and eigenvalue

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} \\ &= H_{\text{Real}}(e^{j\omega}) + jH_{\text{Imag}}(e^{j\omega}) \\ &= |H(e^{j\omega})|e^{j\angle H(e^{j\omega})} \end{aligned} \quad (17)$$

where  $H(e^{j\omega})$  is the **frequency response** that characterizes the system.

From Eq. 16 and Eq. 17, we have:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{+\infty} h[k]e^{j\omega(n-k)} \\ &= e^{j\omega n} \sum_{k=-\infty}^{+\infty} \underbrace{h[k]e^{-j\omega k}}_{H(e^{j\omega})} \end{aligned}$$

The concept of the frequency response of LTI systems is essentially the same for CT and DT systems, except for the fact that the frequency response  $H(e^{j\omega})$  of DT LTI systems is always a periodic function of the frequency variable  $\omega$  with period  $2\pi$ .

To show this distinction to CT systems, we substitute  $\omega + 2\pi$  into Eq. 17, to get

$$H(e^{j(\omega+2\pi)}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j(\omega+2\pi)n}$$

then, using the fact that  $e^{\pm j2\pi n} = 1$ , for any integer  $n$  (i.e., integer multiple of  $2\pi$ ), we see that

$$e^{-j(\omega+2\pi)n} = e^{-j\omega n}e^{-j2\pi n} = e^{-j\omega n} \cdot 1 = e^{-j\omega n}$$

and therefore

$$H(e^{j(\omega+2\pi)}) = H(e^{j\omega})$$

for all  $\omega$ , and more generally,

$$H(e^{j(\omega+2\pi r)}) = H(e^{j\omega})$$

for any integer  $r$ . The sequences

$$\{e^{j\omega n}\} \equiv \{e^{j(\omega+2\pi)n}\}$$

for  $-\infty < n < \infty$  are indistinguishable. This is the reason for the periodicity in DT – the system must respond identically to both input sequences. Thus, we only need to specify  $H(e^{j\omega})$  over *one interval*.

With the linear combination, we have (as in CT Eq. 14, but changed  $x(t)$  for  $x[n]$ ),

$$x[n] = \sum_{k=-\infty}^{\infty} C_k e^{jk\omega_0 n}$$

and replace  $\omega_0$  with  $(2\pi/N)$ , to get

$$x[n] = \sum_k C_k e^{jk(2\pi/N)n}$$

Note again that there are only  $N$  *distinct* signals in this set, and we can change  $k$  for any whole multiple of  $N$  ( $k + N$ ,  $k + 2N$ , etc.), and get the identical sequence (as opposed to the CT case). Therefore we only need to go over *one period* of  $k$ . For example

$$x[n] = \sum_{k=-N_1}^{N_1} C_k e^{jk(2\pi/N)n}$$

where,  $N = 2N_1 + 1$ . Or, more generally, denoting range  $\langle N \rangle$  of one period (e.g.,  $k = \langle N \rangle = 0, 1, 2, \dots, N - 1$  or  $k = \langle N \rangle = 3, 4, \dots, N + 2$ ),

**Synthesis Equation**

$$x[n] = \sum_{k=\langle N \rangle} C_k e^{jk(2\pi/N)n} \quad (18)$$

$$C_k = C_{-k}^*$$

**Analysis Equation**

$$C_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n} \quad (19)$$

for  $x[n]$  periodic with  $N$ .

Note that these equations represent *finite geometric series*.

☞ **Example:** Consider the signal

$$x[n] = \sin(\omega_0 n)$$

which is periodic only if  $\frac{2\pi}{\omega_0}$  is an integer or a ratio of integer. When it is an integer  $N$ , then  $\omega_0 = 2\pi/N$  and thus

$$x[n] = \sin\left(\frac{2\pi}{N}n\right)$$

Expanding (with Euler's relation), we get

$$\begin{aligned} x[n] &= \frac{1}{2j} \left[ e^{j(2\pi/N)n} - e^{-j(2\pi/N)n} \right] \\ &= \frac{1}{2j} e^{j(2\pi/N)n} - \frac{1}{2j} e^{-j(2\pi/N)n} \\ &= \sum_{k=\langle N \rangle} C_k e^{j(2\pi/N)n} \end{aligned}$$

where, clearly  $C_1 = \frac{1}{2j}$  and  $C_{-1} = -\frac{1}{2j}$  and  $C_k = 0$  for any other  $k$ . Recall, that these coefficients repeat with period  $N$ , so  $C_{1+N} = \frac{1}{2j}$  and  $C_{-1+N} = -\frac{1}{2j}$ , etc. Thus:

$$C_1 = \dots C_{1-2N} = C_{1-N} = C_1 = C_{1+N} = C_{1+2N} = \dots = \frac{1}{2j}$$

and

$$C_{-1} = \dots C_{-1-2N} = C_{-1-N} = C_{-1} = C_{-1+N} = C_{-1+2N} = \dots = -\frac{1}{2j}$$

 **Example:** Consider the signal

$$x[n] = \sin\left(\frac{2\pi}{10}n\right)$$

where  $N = \frac{2\pi}{\omega_0} = 10$ . From the previous example, we obtain

$k$	$C_k$
...	...
-11	$-\frac{1}{2j}$
-9	$\frac{1}{2j}$
-1	$-\frac{1}{2j}$
1	$\frac{1}{2j}$
9	$-\frac{1}{2j}$
11	$\frac{1}{2j}$
...	...

In the case where  $\frac{2\pi}{\omega_0}$  is an integer ratio, i.e.,

$$\omega_0 = \frac{2\pi M}{N}$$

the above example  $x[n] = \sin(\omega_0 n)$  becomes

$$x[n] = \frac{1}{2j}e^{jM(2\pi/N)n} - \frac{1}{2j}e^{-jM(2\pi/N)n}$$

from which we see that

$$C_M = \frac{1}{2j}, \text{ and } C_{-M} = -\frac{1}{2j}$$

Example: Consider the signal

$$x[n] = 1 + \sin\left(\frac{2\pi}{N}n\right) + 3 \cos\left(\frac{2\pi}{N}n\right) + \cos\left(\frac{4\pi}{N}n + \frac{\pi}{2}\right)$$

similar to one of the previous examples for CT FS. The signal is periodic with period  $N$ . We can expand  $x[n]$  directly to obtain:

$$\begin{aligned} x[n] &= 1 + \frac{1}{2j} [e^{j(2\pi/N)n} - e^{-j(2\pi/N)n}] + \frac{3}{2} [e^{j(2\pi/N)n} + e^{-j(2\pi/N)n}] \\ &\quad + \frac{1}{2} [e^{j(4\pi n/N + \pi/2)} + e^{-j(4\pi n/N + \pi/2)}] \\ &= 1 + \left(\frac{3}{2} + \frac{1}{2j}\right) e^{j(2\pi/N)n} + \left(\frac{3}{2} - \frac{1}{2j}\right) e^{-j(2\pi/N)n} \\ &\quad + \left(\frac{1}{2} e^{j\pi/2}\right) e^{j2(2\pi/N)n} + \left(\frac{1}{2} e^{-j\pi/2}\right) e^{-j2(2\pi/N)n} \end{aligned}$$

And thus, the FS coefficients are:

	$C_k$	$C_{-k}$
$k = 0$	1	
$k = 1$	$\frac{3}{2} + \frac{1}{2j}$	$\frac{3}{2} - \frac{1}{2j}$
$k = 2$	$\frac{1}{2}j$	$-\frac{1}{2}j$

Recall that these coefficients repeat for  $k + N$ ,  $k + 2N$ , etc.

Example: Consider the signal  $x[n]$  with a period of  $N = 5$ :

$$x[n] = \begin{cases} 1 & n = 0, 1, 2 \\ 0 & n = 3, 4 \end{cases}$$

Using the **analysis equation** Eq. 19, we have

$$\begin{aligned} C_k &= \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n} \\ &= \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk(2\pi/N)n} \\ &= \frac{1}{5} \sum_{n=0}^4 x[n] e^{-jk(2\pi/5)n} \\ &= \frac{1}{5} \sum_{n=0}^2 1 \cdot e^{-jk(2\pi/5)n} \\ &= \frac{1}{5} \left[ e^{-jk(2\pi/5)0} + e^{-jk(2\pi/5)1} + e^{-jk(2\pi/5)2} \right] \\ &= \frac{1}{5} \left[ 1 + e^{-jk(2\pi/5)} + e^{-jk(4\pi/5)} \right] \\ &= \frac{1}{5} \left[ e^{-jk(2\pi/5)} \left( e^{jk(2\pi/5)} + 1 + e^{-jk(2\pi/5)} \right) \right] \\ &= \frac{1}{5} \left[ e^{-jk(2\pi/5)} \left( 1 + 2 \cos \left( (2\pi/5)k \right) \right) \right] \\ &= \frac{1}{5} \left( 1 + 2 \cos \left( (2\pi/5)k \right) \right) e^{-jk(2\pi/5)} \end{aligned}$$

and

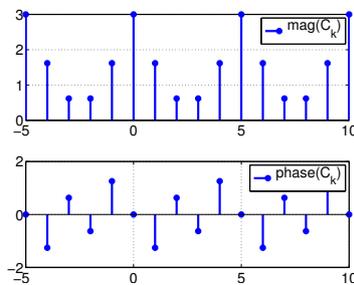
$$C_0 = \frac{1}{5} \sum_{n=0}^2 1 \cdot 1 = \frac{3}{5} = 0.6$$

Thus

$$|C_k| = \left| 1 + 2 \cos \left( \frac{2\pi k}{5} \right) \right|$$

and

$$\angle C_k = \angle \left( \frac{1}{5} \left( 1 + 2 \cos \left( (2\pi/5)k \right) \right) e^{-jk(2\pi/5)} \right)$$



☞ **Example:** Consider the same signal as the previous example, but this time we will work on it from a more generic different point of view.

$$C_k = \frac{1}{N} \sum_{n=0}^{N_p-1} e^{-jk(2\pi/N)n}$$

where  $N_p$  indicates the number of positive (+1) values from  $0 - N_p = 3$  in our particular case; we get

$$\begin{aligned} C_0 &= \frac{1}{N} \sum_{n=0}^{N_p-1} e^{-jk(2\pi/N)n} \\ &= \frac{1}{N} \sum_{n=0}^{N_p-1} 1 \\ &= \frac{N_p}{N} \end{aligned}$$

$$\begin{aligned} C_k &= \frac{1}{N} \sum_{n=0}^{N_p-1} e^{-jk(2\pi/N)n} \\ &= \frac{1}{N} \sum_{n=0}^{N_p-1} \underbrace{\left( e^{-jk(2\pi/N)} \right)}_{\alpha}^n \\ &= \frac{1}{N} \frac{1 - \alpha^{N_p}}{1 - \alpha} \\ &= \frac{1}{N} \frac{1 - (e^{-jk(2\pi/N)})^{N_p}}{1 - e^{-jk(2\pi/N)}} \\ &= \frac{1}{N} \frac{e^{-jk(2\pi/N \frac{1}{2})N_p} (e^{jk(2\pi/N \frac{1}{2})N_p} - e^{jk(2\pi/N \frac{1}{2})N_p})}{e^{-jk(2\pi/N \frac{1}{2})} (e^{jk(2\pi/N \frac{1}{2})} - e^{jk(2\pi/N \frac{1}{2})})} \\ &= \frac{1}{N} \frac{e^{-jk(\pi/N)N_p} (2j \sin(kN_p\pi/N))}{e^{-jk(\pi/N)} (2j \sin(k\pi/N))} \\ &= \frac{1}{N} \frac{e^{-jk(\pi/N)N_p} (\sin(kN_p\pi/N))}{e^{-jk(\pi/N)} (\sin(k\pi/N))} \\ &= \frac{1}{N} \frac{\sin(kN_p\pi/N)}{\sin(k\pi/N)} e^{-jk(\pi/N)(N_p-1)} \end{aligned}$$

which is in Euler / polar form; and equivalent to previous example for  $N = 5$  and  $N_p = 3$ .

## 4.7 Properties of DT FS and comparison with CT FS

From the **synthesis equation** Eq. 18, we can approximate  $x[n]$ ,

$$\hat{x}[n] = \sum_{k=-M}^M C_k e^{jk(2\pi/N)n}$$

We see that for  $M = 4$  the partial sum exactly equals  $x[n]$ ! Thus, unlike in the CT case, there are no convergence ‘issues’ / Gibbs phenomenon – it converges perfectly.

### Frequency Shifting

$$x[n] = \frac{1}{2j} e^{jM(2\pi/N)n} - \frac{1}{2j} e^{-jM(2\pi/N)n}$$

### Multiplication

$$x[n]y[n] = \sum_{l=\langle N \rangle} C_l D_{k-l}$$

which is basically convolution over one period (the summation variable is restricted to an interval of  $N$  consecutive samples)!

### Parseval’s Relation

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |C_k|^2$$

Establishes that the average power of a periodic signal is the same as the sum of the average powers of all the harmonic components.

## A Appendix: Math Review

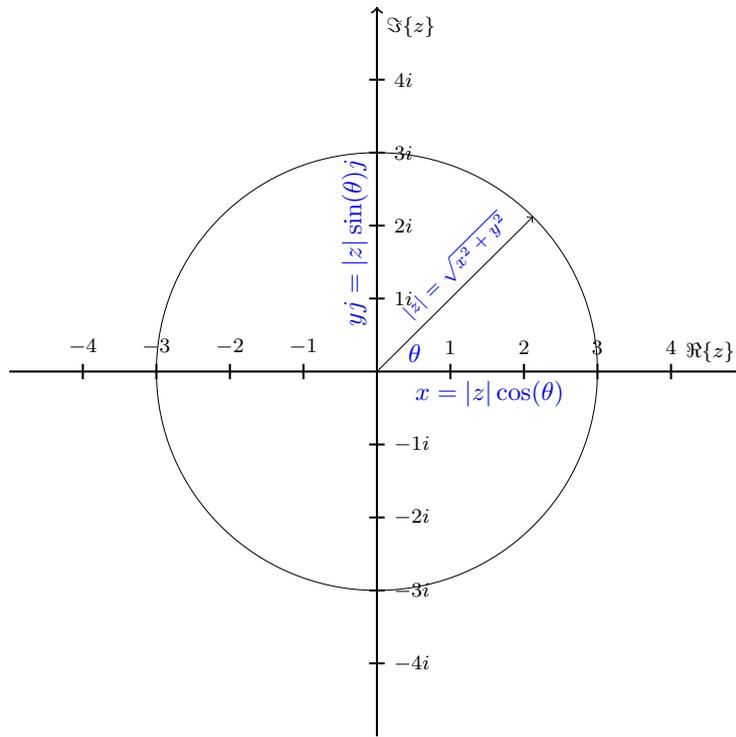


Figure 8: The complex plane showing Euler's formula:  $z = |z|e^{i\phi(z)}$ .

## A.1 Complex Numbers

Complex number, in **Cartesian** form:

$$z = x + iy$$

(where  $i$  is imaginary) i.e.,  $z = \text{Re}(z) + i\text{Im}(z)$ ; in **polar form** (and **Euler form**):

$$\begin{aligned} z &= \underbrace{(r \cos \theta)}_x + \underbrace{(r \sin \theta)}_y i \\ &= |z|e^{i\phi(z)} \\ &= r(\cos \theta + i \sin \theta) \\ &= re^{i\theta} \end{aligned}$$

by **Euler's formula**, where

- $r = |z| = |x + yi| = \sqrt{x^2 + y^2}$  is the **magnitude** of  $z$
- $\theta = \angle z = \phi(z) = \text{atan2}(y, x)$  is the **angle** or **phase** of  $z$

Note that  $i^0 = 1, i^1 = i, i^2 = -1, i^3 = -i$  and thereafter repeating

For example:  $1.14 + 1.64i$ . Can plot, where  $x$  axis is the real part and the  $y$  is the imaginary axis. See Fig. 8.

### Multiplying complex numbers

$$(1 + i)(2 + 3i) = 2 + 5i + 3i^2 = -1 + 5i$$

When you multiply two complex numbers, their magnitudes multiply:

$$|xy| = |x||y|$$

and their phases add:

$$\phi(xy) = \phi(x) + \phi(y)$$

### Dividing complex numbers

1. find **conjugate** of denominator
2. multiply numerator and denominator by this
3. simplify (recall  $i^2 = -1$ , etc.)

$$\frac{3 + 4j}{1 - 2j} = \frac{3 + 4j}{1 - 2j} \cdot \frac{1 + 2j}{1 + 2j} = 5 - 6j$$

Note that the **conjugate** of  $z = x + iy = |z|(\cos \theta + i \sin \theta) = re^{i\theta}$  is

$$\bar{z} = x - iy = r(\cos \theta - i \sin \theta) = re^{-i\theta}$$

an alternative notation is  $z^*$ . On the **complex plane**,  $z$  is reflected across the real axis (hence  $x$  stays the same but  $iy$  reverses its sign).

And this is why the magnitude of a complex number is the amplitude:

$$\begin{aligned} |z| &= \sqrt{x^2 + y^2} \\ &= \sqrt{\underbrace{(r \cos^2 \theta)}_{x^2} + \underbrace{(r \sin^2 \theta)}_{y^2}} \\ &= r \frac{1}{2} + r \frac{1}{2} \\ &= r \end{aligned}$$

## A.2 Euler's formula

By **Euler's formula**<sup>2</sup> we encode complex numbers with exponential notation,

### Euler's Formula

$$e^{j\theta} = \cos \theta + j \sin \theta$$

where  $e^{j\theta}$  is the vector of **magnitude**  $A$  and **phase**  $\theta$ . With magnitude  $A$ , we have

$$Ae^{j\theta} = A \cos \theta + A \sin \theta j$$

The **conjugate** is

### conjugate

$$e^{-j\theta} = \cos \theta - j \sin \theta$$

and from these we can derive:

### Euler's relations

$$\cos \theta = \operatorname{Re}\{e^{j\theta}\} = \frac{1}{2}[e^{j\theta} + e^{-j\theta}]$$

$$\sin \theta = \operatorname{Im}\{e^{j\theta}\} = \frac{1}{2j}[e^{j\theta} - e^{-j\theta}]$$

Note, we can rewrite the sinusoid (in terms of periodic complex exponentials of the same fundamental period) as:

$$A \cos(\omega_0 t + \phi) = \frac{A}{2} e^{j\phi} e^{j\omega_0 t} + \frac{A}{2} e^{-j\phi} e^{-j\omega_0 t}$$

setting  $\phi = 0$ , we can get to

$$\begin{aligned} A \cos(\omega_0 t + 0) &= \frac{A}{2} e^{j0} e^{j\omega_0 t} + \frac{A}{2} e^{-j0} e^{-j\omega_0 t} \\ &= \frac{A}{2} e^{j\omega_0 t} + \frac{A}{2} e^{-j\omega_0 t} \\ &= \frac{A}{2} [e^{j\omega_0 t} + e^{-j\omega_0 t}] \end{aligned}$$

and setting  $A = 1$ , we get

$$\cos(\omega_0 t) = \frac{1}{2} [e^{j\omega_0 t} + e^{-j\omega_0 t}]$$

---

<sup>2</sup>Euler's formula can be derived from expanding  $e^{i\theta}$  with the Taylor series, and collecting imaginary and real parts together

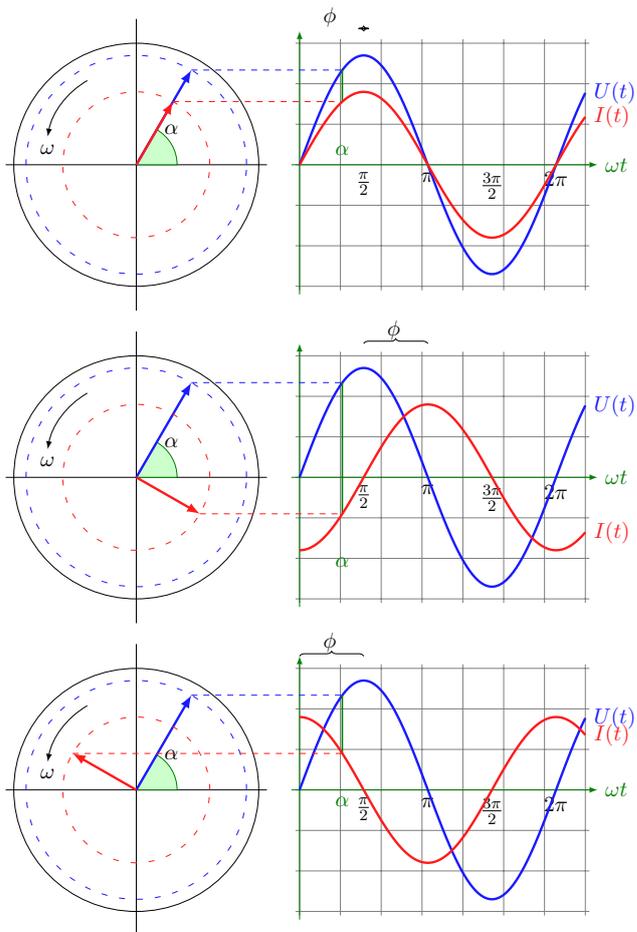


Figure 9: The complex plane, sinusoids, and Euler's formula. Source: <http://www.texample.net/tikz>

### A.3 Geometric Series

#### Finite geometric series

$$\sum_{n=0}^N \alpha^n = \frac{1 - \alpha^{N+1}}{1 - \alpha}$$

#### Infinite geometric series

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1 - \alpha}$$

for  $|\alpha| < 1$  (otherwise it blows up).

#### Deriving the finite series

$$S = \sum_{n=0}^N x^n$$

$$S = 1 + x + x^2 + x^3 + x^4 \dots + x^{N-1} + x^N$$

$$Sx = 1x + xx + x^2x + x^3x + x^4x \dots + x^{N-1}x + x^Nx$$

$$Sx = x + x^2 + x^3 + x^4 + x^5 \dots + x^{N-1} + x^N + x^{N+1}$$

Subtracting,

$$S - Sx = 1 - x^{N+1}$$

so

$$S(1 - x) = 1 - x^{N+1}$$

and thus,

$$S = \frac{1 - x^{N+1}}{1 - x}$$

Now, note that, as  $N \rightarrow \infty$ ,  $x^{N+1} \rightarrow 0$  (recall,  $|x| < 1$ ), so we can replace it with 0, and thus get:

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1 - x}$$

## A.4 Taylor Series Expansion

The **Taylor series** of a real or complex-valued function  $f(x)$  that is infinitely differentiable, for example  $f(x) = e^x$  (we can differentiate this  $\frac{d}{dx}e^x$  over and over again) or  $\frac{d}{dx}\sin(x^2)$ .

It is can be used to approximate functions that do not have a closed form solution; particularly the **transcendental functions** such as  $e^x$ ,  $\sin x$ , and  $\cos x$ .

The steps, for function  $f(x)$ , are

1. Write formula for  $f^{(k)}(x)$  where  $k$  is the number of times to take the derivative
2. Choose constant  $c$  (if not already specified) e.g.,  $c := 0$
3. Write out summation, for  $k = 0, 1, 2, \dots$ ,

$$f(x) \approx f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \dots \quad (20)$$

and calculate coefficients until a pattern emerges

4. Compact it to sigma notation:

$$f(x) \approx \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x - c)^k$$

☞ **Example:** Taylor series expansion of the **exponential function**,

$$f(x) = e^x$$

Now we take Eq. 20. We choose  $c = 0$ . Note that  $f'(0) = 1$  (since in  $\frac{d}{dc}e^c = 1$  for  $c = 0$ , because  $\frac{d}{dc}e^0 = e^0 = 1$ ). This means that all  $f^{(k)}(c)$  for any  $k$  goes to 1; thus Eq. 20 becomes

$$\begin{aligned} e^x &\approx 1 + 1 \cdot x + 1 \cdot \frac{(x - 0)^2}{2!} + 1 \cdot \frac{(x - 0)^3}{3!} + \dots + 1 \cdot \frac{(x - 0)^k}{k!} \\ &\approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^k}{k!} \\ &\approx \sum_{k=1}^{\infty} \frac{x^k}{k!} \end{aligned}$$

With with constant  $c$ , this is the **Maclaurin series**:

$$1 + x + x^2 + x^3 + \dots$$

The idea is that the error term  $\rightarrow 0$  as  $k \rightarrow \infty$

Example: Taylor series expansion of the **sine function**,

$$\begin{aligned}\sin(x) &\approx \sin(c) + \sin'(c) \cdot x + \frac{\sin''(c)}{2!} \cdot x^2 + \frac{\sin'''(c)}{3!} \cdot x^3 + \dots \\ &\approx \sin(0) + \cos(0) \cdot x + \frac{-\sin(0)}{2!} \cdot x^2 + \frac{-\cos(0)}{3!} \cdot x^3 + \dots \\ &\approx 0 + 1 \cdot x + \frac{0}{2!} \cdot x^2 + \frac{-1}{3!} \cdot x^3 + \dots \\ &\approx 0 + x + 0 + \frac{-1}{3!} \cdot x^3 + \dots \\ &\approx x - \frac{x^3}{3!} + \dots\end{aligned}$$