Spectral Methods for Testing Membership in Certain Post Classes and the Class of Forcing Functions

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Background

- Forcing functions represent an important class of functions
- They have been extensively studied in the context of random Boolean networks
- They are known to prevent chaotic behavior in Boolean networks
- There is also evidence that forcing functions are abundantly utilized in transcriptional regulation in eukaryotes (Harris *et al*., 2002)
- Forcing functions have also been used to study certain properties of digital filters, such as convergence, root signals, associative memory, etc.

Background (2)

- Forcing functions exhibit a close relationship to certain function classes that are closed under composition (Post classes)
- These Post functions classes have been applied, e.g., for synthesis and reliability of control systems
- They are also related to the phase transition between order and chaos in random Boolean networks (Shmulevich *et al*., 2003)

Motivation

- ▷ The properties of the above function classes are been studied intensively
- ▷ It is preferable to have efficient membership testing methods
- ▷ A mathematically elegant solution to testing membership in all Post classes have been introduced in (Levchenkov, 2000)
- ▷ Our goal is to develop efficient spectral-type of membership testing methods

Forcing Functions

- **Definition 1:** A Boolean function $f : \{0,1\}^n \rightarrow \{0,1\}$ is said to be forcing if there exists an $i \in \{1,\ldots,n\}$ and $u, v \in \{0,1\}$ such that for all $x_1,\ldots,x_n \in \{0,1\}$, if $x_i = u$ then $f(x_1,\ldots,x_n) = v$
- For example, the function

$$f(x_1, x_2, x_3) = \overline{x}_1 + x_2 x_3$$

is forcing, since if $x_1 = 0$ then $f(x_1, x_2, x_3) = 1$

Testing the Forcing Property

- Let $\mathbf{R}_n(0,1)$ be the Rademacher $(0,1)$ matrix of order n, whose rows correspond to the all n-element binary vectors in lexicographical order
- As an example

$$\mathbf{R}_3^T(0,1) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Testing the Forcing Property (2)

- **Definition 2:** The *forcing transform* of the truth table $\mathbf{f}$ (column vector) of a Boolean function f is

$$\mathbf{c}^{(1,1)} = \mathbf{R}_n^T(0,1) \cdot \mathbf{f}$$

and other derived quantities are

$$\begin{aligned}
\mathbf{c}^{(0,1)} &= |\mathbf{f}| \cdot \mathbf{1} - \mathbf{c}^{(1,1)} \\
\mathbf{c}^{(0,0)} &= \left(2^{n-1} - |\mathbf{f}|\right) \cdot \mathbf{1} + \mathbf{c}^{(1,1)} \\
\mathbf{c}^{(1,0)} &= 2^{n-1} \cdot \mathbf{1} - \mathbf{c}^{(1,1)}
\end{aligned}$$

Testing the Forcing Property (3)

- **Proposition 1:** The function f is a forcing function if and only if there exist $u, v \in \{0, 1\}$ and $i \in \{1, \ldots, n\}$, such that $\mathbf{c}_i^{(u,v)} = 2^{n-1}$. In that case, u is the forcing value of f for variable x_i and v is its forced value

Testing the Forcing Property (3)

- **Proposition 1:** The function f is a forcing function if and only if there exist $u, v \in \{0,1\}$ and $i \in \{1,\ldots,n\}$, such that $\mathbf{c}_i^{(u,v)} = 2^{n-1}$. In that case, u is the forcing value of f for variable x_i and v is its forced value
- Information about all forcing variables, forcing values, and forced values is contained in the vectors $\mathbf{c}^{(u,v)}$
- Example: The truth table of the function $f(x_1, x_2, x_3) = \overline{x}_1 + x_2 x_3$ is $\mathbf{f} = [1,1,1,1,0,0,0,1]^T$, and

$$\begin{aligned}\mathbf{c}^{(1,1)} &= \begin{bmatrix} 1 & 3 & 3 \end{bmatrix}^T, \quad \mathbf{c}^{(0,1)} = \begin{bmatrix} 4 & 2 & 2 \end{bmatrix}^T \\ \mathbf{c}^{(1,0)} &= \begin{bmatrix} 3 & 1 & 1 \end{bmatrix}^T, \quad \mathbf{c}^{(0,0)} = \begin{bmatrix} 0 & 2 & 2 \end{bmatrix}^T\end{aligned}$$

Certain Post Functions

- **Definition 3:** A Boolean function f belongs to class A^{μ}, $\mu \geq 2$, if any μ vectors on which the function takes on the value 1 have a common component equal to 1 (some of these μ vectors may be repeated)
- For example, the function

$$f\left(x_1, x_2, x_3\right) = x_1 x_2 + x_2 x_3 + x_1 x_3 \tag{1}$$

is A^2, since any 2 vectors on which the function is equal to 1 have a common unity component

Testing the A^μ Property

- **Definition 4:** Define a matrix $\mathbf{B}_{n,\mu}$ whose rows contain all indicator functions of sets that are not μ-inseparable (excluding the empty set)
- For example, for $n = 3$ and $\mu = 2$,

$$\mathbf{B}_{3,2} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

- **Definition 5:** The A^μ *transform* is

$$\mathbf{c}_{n,\mu} = \mathbf{B}_{n,\mu} \cdot \mathbf{f}$$

Testing the A^μ Property (2)

- **Proposition 2:** Let $\mu \geq 2$ be given. Let $f : \{0,1\}^n \rightarrow \{0,1\}$ be a Boolean function that is equal to 0 on the all-zero vector and contains exactly $m \geq 2$ ones in its truth table $\mathbf{f}$. Let $k = \min(\mu, m, n)$. Then, f is A^μ if and only if $\mathbf{c}_{n,k}$ does not contain an element equal to k.

Testing the A^{μ} Property (2)

- **Proposition 2:** Let $\mu \geq 2$ be given. Let $f : \{0,1\}^n \longrightarrow \{0,1\}$ be a Boolean function that is equal to 0 on the all-zero vector and contains exactly $m \geq 2$ ones in its truth table $\mathbf{f}$. Let $k = \min(\mu, m, n)$. Then, f is A^{μ} if and only if $\mathbf{c}_{n,k}$ does not contain an element equal to k.
- Example: The truth table of the function $f(x_1, x_2, x_3) = x_1 x_2 + x_2 x_3 + x_1 x_3$ is $\mathbf{f} = [0,0,0,1,0,1,1,1]^T$, $k = \min(\mu, m, n) = \min(2,4,3) = 2$, and

$$\mathbf{c}_{3,2} = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}^T$$

Extensions

- The same results also apply to the class A^{∞}
- Methods for the function classes a^{μ} and a^{∞} are obtained by duality
- The methods can be extended to so-called generalized forcing functions
- Membership in the class of nested canalizing (nested forcing) functions can also be tested
- Spectral methods for testing several other (Post) function classes, such as monotone, self-dual and linear, are described in (Agaian *et al*., 1995)

References

- ▷ Agaian, S., Astola, J. and Egiazarian, K. (1995) *Binary Polynomial Transforms and Nonlinear Digital Filters*, Marcel Dekker.
- ▷ Harris, S. E., Sawhill, B. K., Wuensche, A. and Kauffman, S. (2002) A model of transcriptional regulatory networks based on biases in the observed regulation rules. *Complexity*, Vol. 7, No. 4, pp. 23–40.
- ▷ Levchenkov, V. S. (2000) Boolean equations and Post classes. *Doklady Akademii Nauk*, Vol. 373, No. 3, pp. 316–319, (in Russian).
- ▷ Shmulevich, I., Lähdesmäki, H., Dougherty, E. R., Astola, J. and Zhang, W. (2003) The role of certain Post classes in Boolean network models of genetic networks. *Proceedings of the National Academy of Sciences of the USA*, Vol. 100, No. 19, pp. 10734–10739.
- ▷ Shmulevich, I., Lähdesmäki, H. and Egiazarian, K. (2004) Spectral methods for testing membership in certain Post classes and the class of forcing functions. *IEEE Signal Processing Letters*, Vol. 11, No. 2, pp. 289–292.