Online Entropy Estimator

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Outline

Introduction

Estimator for Unknown Entropy Sources

New Estimator

Empirical Results

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Introduction
PRNG with Entropy Input

- Pseudo Random Number Generator with entropy sources

- PRNG is deterministic: A specific state will always produce the same output
- Uncertainty come from entropy sources
- When "enough uncertainty is collected" reseed the state
  - How do we know when to reseed?

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Entropy - Information Theoretical Model

- **Entropy Source**

- Outputs element from **sample space** $\mathcal{X}$

- **Concrete output** at time $t$: $x_t \in \mathcal{X}$

- Source represents sequence of identical and independent distributed **random variables** $X_1, X_2, \ldots$

  $$\Pr(X_t = \eta) = p_\eta \text{ for all } t, \ p = (p_\eta)_{\eta \in \mathcal{X}}$$

- **Sequence of random variables** $X_{[t_1, t_2]} = X_{t_1}, X_{t_1+1}, \ldots, X_{t_2}$

- **Sequence of concrete outputs** $x_{[t_1, t_2]} = x_{t_1}, x_{t_1+1}, \ldots, x_{t_2}$

- **Empirical distribution** of $x_{[1, n]}$: $\hat{p}_\eta = \frac{\# \{ t : x_t = \eta \}}{n}$
Entropy - Definitions

- **Shannon entropy**: Measure of average number of binary questions before guessing the output
  \[ H(p) = - \sum_{\eta \in \mathcal{X}} p_\eta \log_2 p_\eta \]

- **Rényi entropy**: Measure of correlation probability
  \[ H_\alpha(p) = \frac{1}{1 - \alpha} \log_2 \left( \sum_{\eta \in \mathcal{X}} p_\eta^\alpha \right) \]

- **Min entropy**: Lower bound for all entropy measures
  \[ H_\infty(p) = - \log_2 \left( \max_{\eta \in \mathcal{X}} p_\eta \right) \]

- **Relation** for \( \alpha > 1 \):
  \[ H_\infty(p) \leq H_\alpha(p) \leq H(p) \leq \log_2 |\mathcal{X}| \]
Estimator for Unknown Entropy Sources
Requirements

- Estimator $\hat{H}$ should:
  - Work with **unknown sources**
  - Be **pessimistic** $E(\hat{H}) \leq H(p)$
  - Be **efficient**
  - Given an estimate for **each output**
    - Want $\hat{H}$ such that $\frac{1}{n-r} \sum_{t=r+1}^{n} \hat{H}(X_{[t-r,t]}) \xrightarrow{n \to \infty} H(p)$
  - Work with **any source**
**Known Estimators (1)**

- **Plug-in or maximum-likelihood** estimator: use $H(\hat{p})$
  - Estimate only for **whole data set**
  - Need counter for each $\eta \in \mathcal{X}$
  - Can be applied on a data window but still not efficient

- **Compression** based on **frequency counting**:
  e.g. Huffman coding, arithmetic coding
  - **Not efficient** (Huffman coding need tree of size $|\mathcal{X}|$)
Known Estimators (2)

- **Compression** based on **match length** [Lempel Ziv]

\[ L_t^r(x) = 1 + \max_{t-r \leq j \leq t-1} \{ k : x_{[t,t+k-1]} = x_{[j,j+k-1]} \} \]

- **Classical estimator:**

\[ \frac{L_t^r(X)}{\log_2 r} \to \frac{1}{H} \quad \text{a.s.} \quad (r \to \infty) \]

- Estimate only for **whole data set**
- Need **future values**, no upper bound for \( L_t^r(x) \)
Known Estimators (3)

- **LZ estimator** with intermediate values, 
  \[ \hat{H}_{LZ}^r(x_{[t,n]}) = \log_2 \frac{r}{L_t(x)} \]

  \[ \frac{1}{n - r} \sum_{t=r+1}^{n} \hat{H}_{LZ}^r(x_{[t,n]}) \rightarrow H \text{ a.s. } (n, r \rightarrow \infty) \]

- Gives an estimate for **each** \( t \geq r \)
- Need **future values**, no upper bound for \( L_t^r(x) \)
Known Estimators (4)

- Based on **transition frequencies** [Bucci Luzzi 2005]
- **Count transitions** from 0 to 1 or from 1 to 0
- **Expected number of transition** in \( n + 1 \) bits: \( n2p_0(1 - p_0) \)
- Use: \(-\log_2 y \geq \frac{1}{\ln(2)}(1 - y)\) for \( 0 < y < 1 \)
- **Entropy:**

\[
-\rho_0 \log_2 \rho_0 - (1 - \rho_0) \log_2 (1 - \rho_0) \geq \frac{1}{\ln(2)}2\rho_0(1 - \rho_0)
\]

- **Only binary sources**

- **Our idea:** Extend to **non-binary case**
New Estimator
Idea

- **Number of comparisons** before finding **last occurrence** of **current element**:

\[
el_t^r = \begin{cases} 
  r & \text{if } x_t \neq x_{t-j}, 1 \leq j \leq r, \\
  \min \{0 \leq j \leq r - 1 : x_{t-1-t} = x_t\} & \text{otherwise.}
\end{cases}
\]

- **Estimator**:

\[
\hat{H}_{pv}^r(x_{[t-r,t]}) = \frac{1}{\ln(2)} \sum_{j=1}^{\ell_t^r} \frac{1}{j}
\]
Expected value: $E \left( \hat{H}_{pv}^r(X_{[t-r,t]}) \right) = \frac{1}{\ln(2)} \sum_{\eta \in \mathcal{X}} p_{\eta} \sum_{i=1}^{r} \frac{(1 - p_{\eta})^i}{i}$

Using results on (r+1)-dependent random variables:

$$\frac{1}{n-r} \sum_{t=r+1}^{n} \hat{H}_{pv}^r(x_{[t,n]}) \rightarrow E \left( \hat{H}_{pv}^r(X_{[t-r,t]}) \right) \text{ a.s. (n} \rightarrow \infty)$$

Taylor series of logarithm ($0 < x < 1$):

$$\ln \frac{1}{x} = \sum_{i=1}^{\infty} \frac{(1 - x)^i}{i} \geq \sum_{i=1}^{r} \frac{(1 - x)^i}{i}$$

Lower bound for entropy:

$$E \left( \hat{H}_{pv}^r(X_{[t-r,t]}) \right) \leq \sum_{\eta \in \mathcal{X}} p_{\eta} \log_2 \frac{1}{p_{\eta}} = H(p)$$
Empirical Results
Test Data

- Input data for **Linux random number generator**
  - Cooperation with Lacharme, Strubel, Videau
  - Time between interrupts from **user input (mouse, keyboard)**
  - Time measured in **jiffies** and **cycles**

- Data from **iPhone GPS** device
  - Cooperation with Lauradoux, Ponge
  - Measurement of altitude, longitude, latitude, acceleration and compass (heading)
  - **Indoor** (less movement) and **outdoor** (more movement) measurements

- Compare to **LZ estimator**
Linux RNG - Cycle Count Difference

Entropy

Cycles Differences

Number of comparisons

- LZ estimator has complexity as good as our estimator
Linux RNG - Jiffies Count Difference

**Entropy**

- Shannon entr
- Renyi entr
- Min entr
- New estim, $r = 1$
- New estim, $r = 2$
- Lz estim, $r = 2$
- New estim, $r = 10$
- New estim, $r = 100$
- Lz estim, $r = 10$
- New estim, $r = 100$
- Lz estim, $r = 100$

**Number of comparisons**

- Better complexity than LZ estimator
Extreme case for LZ estimator if source has (almost) no entropy
Estimated entropy of new estimator **not far from LZ estimate**
Better complexity than LZ estimator
Conclusion
Conclusion

- Entropy estimator for unknown entropy sources with changing behavior
- At most $r$ comparisons
- For independent sequences the expected value of the estimator is lower bound for entropy
- Gives estimate for each output value

- Only Shannon entropy, not Rényi or Min entropy
- Proof not for correlated sources
- Estimates lower bound, exact value only for $r \to \infty$