Aalto University
School of Science

Links Between Truncated Differential and Multidimensional Linear Properties of Block Ciphers and Underlying Attack Complexities

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Outline

Statistical Attacks
- Truncated Differential (TD) Cryptanalysis
- Multidimensional Linear (ML) Cryptanalysis

Link between ML and TD Attacks
- Mathematical Relation between ML and TD
- Complexity of TD and ML Distinguishing Attacks

Statistical Saturation Attack
- Definition
- Statistical Saturation Attack on PRESENT

Converting a ML Attack to a TD Attack
- Example on PRESENT
- Conclusion

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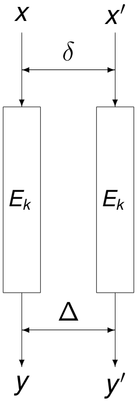
Differential Cryptanalysis [Biham Shamir 90]

Difference between plaintext and ciphertext pairs



Input difference : δ
Output Difference : Δ

Differential Probability :

$$\mathbf{P}[\delta \to \Delta] = P_x[\, E_k(x) \oplus E_k(x \oplus \delta) = \Delta]$$

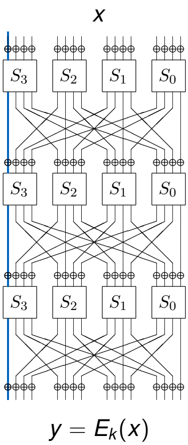
Truncated Differential (TD) [Knudsen 94] :

Set of input differences : $\delta \in A$
Set of output differences : $\Delta \in B$

$$\mathbf{P}[A \to B] = \frac{1}{|A|} \sum_{\delta \in A} \sum_{\Delta \in B} P[\delta \to \Delta]$$

Linear Cryptanalysis [Tardy Gilbert 91] [Matsui 93]

Linear relation involving plaintext, key and ciphertext bits



Input mask : u

Output mask : v

Correlation :

$$\mathbf{cor}_x(u, v) = 2 \cdot P_x \left[u \cdot x \oplus v \cdot E_k(x) = 0\right] - 1$$

Multidimensional Linear (ML) Approximation [Hermelin et al 08] :

Set of masks $(u, v) \in U \times V \setminus \{0, 0\}$

Capacity :

$$C = \sum_{u \in U \setminus \{0\}} \sum_{v \in V \setminus \{0\}} \mathbf{cor}_x^2(u, v)$$

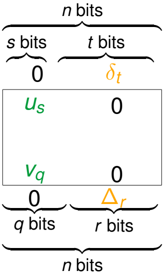
Outline

Link between Differential and Linear Cryptanalysis

[Chabaud Vaudenay 94] :

Let $F : \mathbb{F}_2^n \to \mathbb{F}_2^m$

$$\mathbf{P}[\delta \to \Delta] = 2^{-m} \sum_{u \in \mathbb{F}_2^n} \sum_{v \in \mathbb{F}_2^m} (-1)^{u \cdot \delta \oplus v \cdot \Delta} \mathbf{cor}_x^2(u, v)$$



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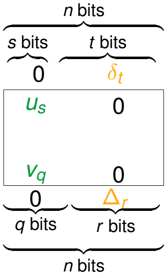

Generalization :

- ML : $[(u_s, 0), (v_q, 0)]_{u_s \in \mathbb{F}_2^s \setminus \{0\},\ v_q \in \mathbb{F}_2^q}$ with capacity C
- TD : $[(0, \delta_t), (0, \Delta_r)]_{\delta_t \in \mathbb{F}_2^t,\ \Delta_r \in \mathbb{F}_2^r}$ with probability p

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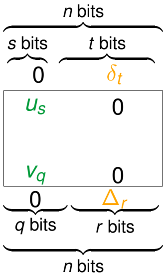

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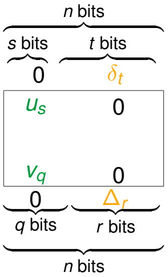

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- TD is a chosen plaintext (CP) attack
- ML is a known plaintext (KP) attack

Data Complexity of a Distinguishing Attack

[Selçuk 07] $P_S = 50\%$ and $\varphi_a = \Phi^{-1}(1 - 2^{-a})$, with a the advantage

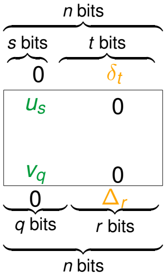
- Multidimensional Linear :

$$N^{ML} = \frac{2^{(s+q+1)/2}}{C} \cdot \varphi_a$$

- Truncated Differential :

$$N^{TD} = \frac{2^{-q+1}}{M \cdot (p - 2^{-q})^2} \cdot \varphi_a^2,$$

where M is the size of a structure (usually $M = 2^t$)



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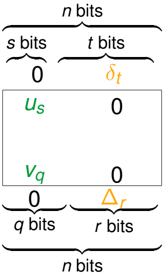

- For $p = 2^{-q}(C + 1)$:

$$N^{TD} = \frac{2^{q+1}}{2^t \cdot C^2} \cdot {\varphi_a}^2$$

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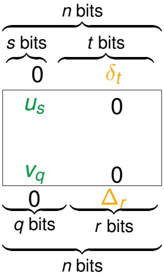

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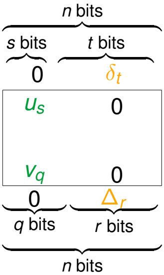

$$N^{TD} = \frac{1}{2^n} \cdot (N^{ML})^2$$

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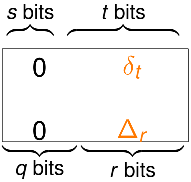
$$\begin{array}{|cc|} \multicolumn{2}{c}{\overbrace{\hspace{8em}}^{n \text{ bits}}} \\ \multicolumn{1}{c}{\overbrace{0}^{s \text{ bits}}} & \multicolumn{1}{c}{\overbrace{\delta_t}^{t \text{ bits}}} \\ \hline u_s & 0 \\ & \\ v_q & 0 \\ \hline \multicolumn{1}{c}{\underbrace{0}_{q \text{ bits}}} & \multicolumn{1}{c}{\underbrace{\Delta_r}_{r \text{ bits}}} \\ \multicolumn{2}{c}{\underbrace{\hspace{8em}}_{n \text{ bits}}} \end{array}$$

$$N^{TD} = \frac{1}{2^n} \cdot (N^{ML})^2$$

$N^{TD} \leq N^{ML}$ with equality when using the full codebook

Truncated Differential Distinguisher



$$x = (x_s, x_t)$$

$$y = (y_q, y_r)$$

M : size of a structure
S : number of structures

$$N^{TD} = S \cdot M$$

$D = 0$
for S values of $x_s \in \mathbb{F}_2^s$ **do**
  Create a table T of size M
  for M values of $x_t \in \mathbb{F}_2^t$ **do**
    $(y_q, y_r) = E((x_s, x_t))$
    $T[x_t] = y_q$
  for all pairs (x_t, x_t') **do**
    if $(T[x_t] \oplus T[x_t']) == 0$ **then**
      $D += 1$

For S structures

For all elements in a structure

Store the partial ciphertexts

Count the number of pairs which have no difference on the q bits

Truncated Differential Distinguisher

Time Complexity : Verifying all pairs

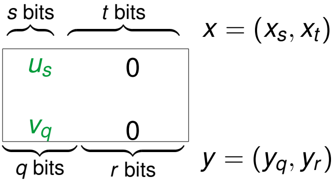
$$Time \approx S \cdot M^2/2$$

Memory Complexity : Storing all ciphertexts inside a structure

$$Memory \approx M$$

$D = 0$	
for S values of $x_s \in \mathbb{F}_2^s$ **do**	For S structures
Create a table T of size M	
for M values of $x_t \in \mathbb{F}_2^t$ **do**	For all elements in a structure
$(y_q, y_r) = E((x_s, x_t))$	Store the partial ciphertexts
$T[x_t] = y_q$	
for all pairs (x_t, x'_t) **do**	Count the number of pairs which have no difference on the q bits
if $(T[x_t] \oplus T[x'_t]) == 0$ **then**	
$D += 1$	

Multidimensional Linear Distinguisher



Set a counter D to 0
Create a table T of size 2^{q+s}
for N^{ML} plaintexts **do**
    $(y_q, y_r) = E((x_s, x_t))$
    $T[(x_s, y_q)] += 1$
for all (x_s, y_q) **do**
    $D += (T[(x_s, y_q)] - N/2^{q+s})^2$

For N^{ML} plaintexts
  Count the number of occurrences of each pair (x_s, y_q)

Compute the statistic

Multidimensional Linear Distinguisher

Time Complexity : Reading all messages

$$Time \approx N^{ML}$$

Memory Complexity : Storing the number of occurrences of $(x_s^i, y_q^j)_{i,j}$

$$Memory \approx 2^{s+q}$$

Set a counter D to 0
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Complexities of TD and ML Attacks

- ML distinguisher :

$$Data = N^{ML}$$

$$Time \approx N^{ML}$$

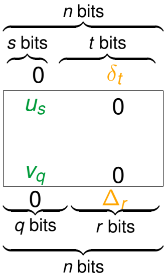
$$Memory \approx 2^{s+q}$$

- TD distinguisher :

$$Data = N^{TD} = S \cdot 2^{t} < N^{ML}$$

$$Time \approx N^{TD} \cdot 2^{t-1}$$

$$Memory \approx 2^{t}$$



Question : Can we decrease the time complexity of a TD attack?

TD with Less Time Complexity

- Dominant part: Verifying the output difference for each pair of ciphertexts

Example :

- 4 ciphertexts : (y_1, b_1) (y_2, b_2) (y_1, b_3) (y_3, b_4)
  1 pair with equal y_i
- Previous algorithm : 6 comparisons

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Improved Version :

- Count the occurrences of each y_i :

	y_1	y_2	y_3
$T[y_i]$	2	1	1

and compute $D = \sum_i T[y_i](T[y_i] - 1)/2 = 1$

TD with Less Time Complexity

$D = 0$
for S values of $x_s \in \mathbb{F}_2^s$ **do**
  Create a table T of size 2^q
  for M values of $x_t \in \mathbb{F}_2^t$ **do**
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For S structures

For all elements in a structure

Count the number of occurrences of the partial ciphertexts

Compute the statistic

$$Data = N^{TD} = S \cdot M < N^{ML}$$

$$Time \approx max(N^{TD}, S \cdot 2^q)$$

$$Memory \approx 2^q$$

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Remark :

This distinguisher is the same as the statistical saturation (SS) distinguisher

Outline

Statistical Saturation (SS) Attack [Collard Standaert 09]

Idea :

- "Dual" of the saturation attack
- Takes advantage of several plaintexts with some fixed bits while the others vary randomly
- We observe the diffusion of the fixed bits during the encryption process

Application on PRESENT [Bogdanov et al 08] :

- Distinguisher on 20 / 21 rounds
- Key-recovery on 24 rounds

Link between SS, TD and ML distinguishers

Link [Leander 11] :
For a fixed $x_s \in \mathbb{F}_2^s$, we denote by $C(x_s)$ the capacity of the distribution of y_q :

$$C = 2^{-s} \sum_{x_s \in \mathbb{F}_2^s} C(x_s)$$

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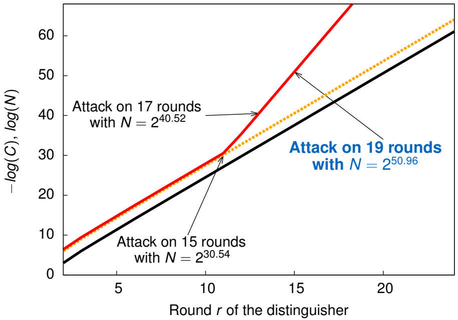
- SS attacks link algorithmically with TD attacks

On the SS Attack on PRESENT [Collard Standaert 09]

Attack on $r + 4$ rounds with $M = 2^{32}$



- [Collard Standaert 09]
  Data increases linearly
- [Leander 11]
  Estimate of the capacity C
- [Our work]
  Data is $N = \dfrac{2^{q+1}}{M \cdot C^2} \cdot \varphi_a^2$

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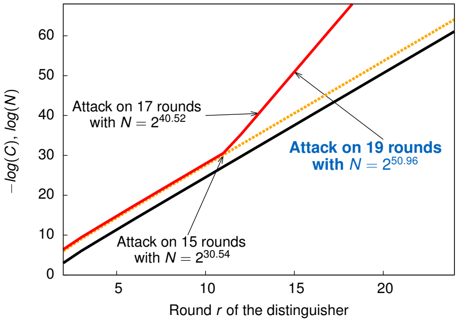

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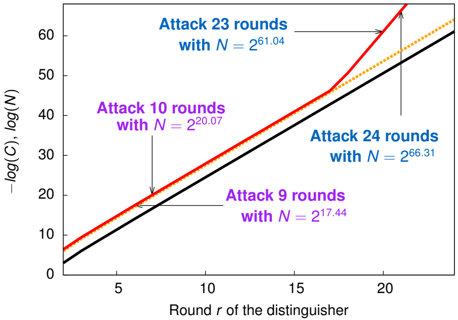
- The attack has been verified experimentally [Kerckhof et al 11]
- Our estimate match with the experiments (N around 2^{51} for 19 rounds)

On the SS Attack on PRESENT [Collard Standaert 09]

Attack on $r + 3$ rounds with $M = 2^{48}$



- Estimate of the capacity C
- *Data* is proportional to $\frac{1}{C}$
- *Data* is $N = \frac{2^{q+1}}{M \cdot C^2} \cdot \varphi_a^2$

- In this model, one can only perform an attack 23 rounds

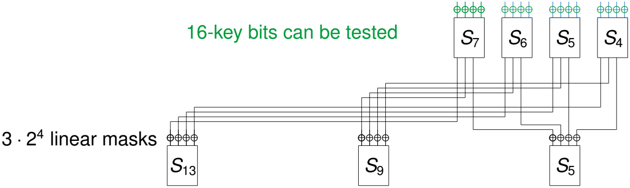
Outline

KP ML and CP TD Attacks : An Example on PRESENT

[Cho 10] :

- ML distinguisher on 24 rounds
- KP ML attack on 26 rounds (inversion of the first and last round)

First round : (In Cho's ML characteristic)



- KP ML $\Rightarrow$ Guess 16-key bits

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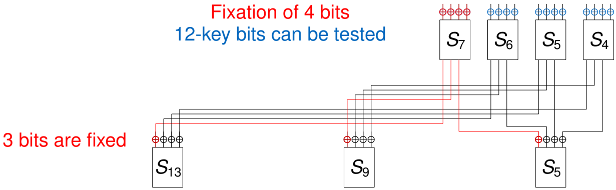


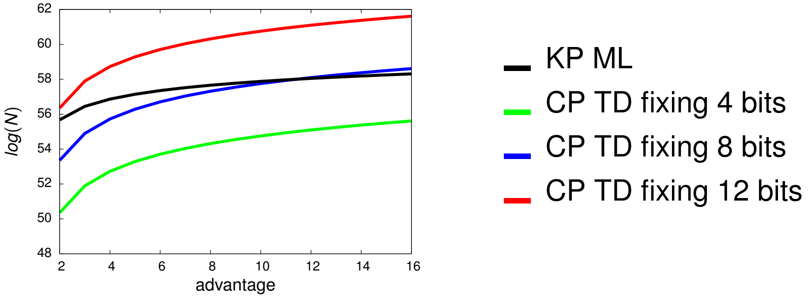
- KP ML $\Rightarrow$ Guess 16-key bits

Using the link between TD and ML

- CP TD $\Rightarrow$ Guess $4, 8, 12, 16$-key bits

Example of CP TD Attack on 24 Rounds of PRESENT

Data Complexity (*Data*) :



- The *Data* of a KP ML is proportional to $\varphi_a = \Phi^{-1}(1 - 2^{-a})$
- The *Data* of a CP TD is proportional to φ_a^2
- Depending of the size of the fixation, the data complexity of a CP ML attack can be smaller than for a KP ML attack

Example of CP TD Attack on 24 Rounds of PRESENT

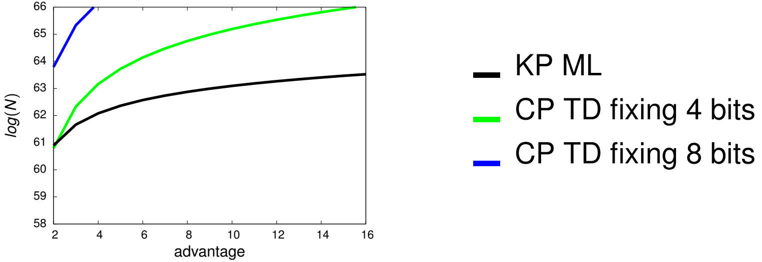
Fixing 4 bits :

Model	a	*Data*	*Memory*	$Time_1$	$Time_2$
CP TD	10	$2^{54.75}$	2^{29}	$2^{54.75}$	2^{70}
KP ML	5	$2^{57.14}$	2^{32}	$2^{57.14}$	2^{75}

$Time_1$: Complexity of the distillation phase
$Time_2$: Complexity of the search phase

- Data, time and memory complexities of the CP TD are smaller than those of a KP ML attack

Example of CP TD Attack on 26 Rounds of PRESENT



Model	a	*Data*	*Memory*	$Time_1$	$Time_2$
CP TD	4	$2^{63.16}$	2^{29}	$2^{63.16}$	2^{76}
KP ML	4	$2^{62.08}$	2^{32}	$2^{62.08}$	2^{76}

- A CP TD attack on 26 rounds of PRESENT with less memory than the KP ML attack
- The previous differential-type attack was on 19 rounds

Conclusion

In this work :

- We analyze the complexities of some statistical attacks and their relation
- We show that the SS attack is a TD attack
- We illustrate that a KP ML attack can be converted to a CP TD attack with smaller complexities

Thank You